Finite Dependence

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- Estimation of dynamic discrete choice models is complicated by the calculation of expected future payoffs.
- These complications are mitigated when finite dependence holds.
- Intuitively, ρ period dependence holds when two sequences of weighted choices leading off from different initial choices generate the same distribution of state variables $\rho + 1$ periods later.
- Most empirical applications have the finite dependence property.
- Under the conditional independence assumption, finite dependence has empirical content (can be tested) without specifying utilities.

- $t \in \{1, \ldots, T\}$ stands for time, where $T \leq \infty$.
- $x_t \in \{1, \ldots, X\} \equiv \mathbb{X}$ is the state at t, where $\mathbb{X}$ is a finite set.
- $j \in \{1, \ldots, J\}$ is a (mutually exclusive) choice.
- $d_{jt} = 1$ if j is picked at t and otherwise $d_{jt} = 0$.
- $f_{jt}(x_{t+1}|x_t)$ is the probability of x_{t+1} occurring in period $t+1$ conditional on x_t and $d_{jt} = 1$.
- $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ has continuous support and is IID over t with PDF $g(\epsilon_t)$ satisfying $E[\max\{\epsilon_{1t}, \ldots, \epsilon_{Jt}\}] \leq \overline{\epsilon} < \infty$.
- For some $\beta \in (0, 1)$, the individual sequentially chooses the vector $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ to maximize:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1}d_{jt}\left[u_{jt}(x_t) + \epsilon_{jt}\right]\right\} \tag{1}$$

- $d_t^o(x_t, \epsilon_t)$ is the optimal decision rule with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$.
- $p_t(x_t) \equiv (p_{1t}(x_t), \ldots, p_{Jt}(x_t))$ are the CCPs, where:

$$p_{jt}(x_t) \equiv \int d_{jt}^o\left(x_t, \epsilon_t\right) g\left(\epsilon_t\right) d\epsilon_t \tag{2}$$

- $V_t(x_t)$ is the ex-ante value function defined as :

$$V_t(x_t) \equiv E\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t} d_{j\tau}^o\left(x_\tau, \epsilon_\tau\right)\left(u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right)\right\}$$

- The conditional value function for action j defined as:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x_{t+1}=1}^{X} V_{t+1}(x_{t+1}) f_{jt}\left(x_{t+1} | x_t\right) \tag{3}$$

- The conditional value function correction for action j is defined:

$$\psi_j[p_t(x)] \equiv V_t(x) - v_{jt}(x) \tag{4}$$

- For all $\rho \leq T - t$ and $\tau = t+1, \ldots, t+\rho$, define any weights $\omega_{k\tau}(t, x_\tau, j)$ satisfying:

$$|\omega_{k\tau}(t, x_\tau, j)| < \infty \text{ and } \sum_{k=1}^{J} \omega_{k\tau}(t, x_\tau, j) = 1$$

.

- We showed $v_{jt}(x_t) =$

$$u_{jt}(x_t) + \sum_{x=1}^{X} \beta^{\rho+1} V_{t+\rho+1}(x) \kappa_{t+\rho+1}(x|t, x_t, j) + \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \begin{bmatrix} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{bmatrix} \omega_{k\tau}(t, x_\tau, j) \kappa_\tau(x_\tau|t, x_t, j) \tag{5}$$

where $\kappa_{t+1}(x_{t+1}|t, x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and:

$$\kappa_{\tau+1}(x_{\tau+1}|t, x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(t, x_\tau, j) f_{k\tau}(x_{\tau+1}|x_\tau) \kappa_\tau(x_\tau|t, x_t, j) \tag{6}$$

Finite Dependence
Definition

- The pair of choices $\{i, j\}$ exhibits *ρ-period dependence* at (t, x_t) if there exist a pair of sequences of decision weights:

$$\{\omega_{k\tau}(t, x_\tau, i)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)} \text{ and } \{\omega_{k\tau}(t, x_\tau, j)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$$

  such that for all $x_{t+\rho+1} \in \{1, \ldots, X\}$:

$$\kappa_{t+\rho+1}(x_{t+\rho+1}|t, x_t, i) = \kappa_{t+\rho+1}(x_{t+\rho+1}|t, x_t, j)$$

- Finite dependence:
  1. trivially holds for $\rho = T - t$ when $T < \infty$, but only merits attention when $\rho < T - t$.
  2. extends to games by conditioning on the player as well.
  3. might hold for some choice pairs but not others, and for certain states but not others.
  4. could be defined for mixed choices to start the sequence, not just deterministic moves; this analysis extends to the more general case.

- If there is finite dependence for (t, x_t, i, j), then:

$$u_{jt}(x_t) + \psi_j[p_t(x_t)] - u_{it}(x_t) - \psi_i[p_t(x_t)] =$$

$$\sum_{(k,\tau,x_\tau)=(1,t+1,1)}^{(J,t+\rho,X)} \beta^{\tau-t} \left\{ \begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right\} \left[\begin{array}{l} \omega_{k\tau}(t, x_\tau, i)\kappa_\tau(x_\tau|t, x_t, i) \\ -\omega_{k\tau}(t, x_\tau, j)\kappa_\tau(x_\tau|t, x_t, j) \end{array} \right] \tag{7}$$

- To derive this equation:
  - appealing to (4), replace $v_{jt}(x)$ with $V_t(x) - \psi_j[p_t(x_t)]$ in (5)
  - form an analogous equation for i
  - difference the two resulting equations and note the $V_t(x)$ terms cancel.

Simple Examples of Finite Dependence

Terminal choices

- Terminal choices and renewal choices are widely assumed in structural econometric applications of dynamic optimization problems and games.
- A *terminal choice* ends the evolution of the state variable with an *absorbing state* that is independent of the current state.
- If the first choice denotes a terminal choice, then:

$$f_{1t}(x_{t+1}|x) \equiv f_{1t}(x_{t+1})$$

for all $(t, x) \in \mathbb{T} \times \mathbb{X}$ and hence:

$$\sum\nolimits_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) = f_{1,t+1}(x_{t+2})$$

- Setting $\omega_{k\tau}(t, x, i) = 0$ for all (x, i) and $k \neq 1$, Equation (7) implies:

$$\begin{aligned} & u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = \; & \sum\nolimits_{x_{t+1}=1}^{X} \beta \left\{ u_{1,t+1}(x_{t+1}) + \psi_1 \left[p_{t+1}(x_{t+1}) \right] \right\} f_{jt}(x_{t+1}|x_t) \end{aligned}$$

Simple Examples of Finite Dependence

Renewal choices

- Similarly a *renewal choice* yields a probability distribution of the state variable next period that does not depend on the current state.
- If the first choice is a renewal choice, then for all $j \in \{1, \ldots, J\}$:

$$
\begin{aligned}
\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) \\
&= f_{1,t+1}(x_{t+2}) \sum_{x_{t+1}=1}^{X} f_{jt}(x_{t+1}|x_t) \\
&= f_{1,t+1}(x_{t+2}) \qquad (8)
\end{aligned}
$$

- In this case Equation (7) implies:

$$
\begin{aligned}
& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\
= & \sum_{x=1}^{X} \beta \{u_{1,t+1}(x) + \psi_1[p_{t+1}(x)]\} [f_{jt}(x|x_t) - f_{1t}(x|x_t)]
\end{aligned}
$$

Simple Examples of Finite Dependence

An example of 2-period finite dependence

- How does finite dependence work when $\rho > 1$?
- Consider the following model of labor supply and human capital.
- In each of T periods an individual chooses:
  - $d_{2t} = 1$ to work
  - $d_{1t} = 1$ to stay home.
- She accumulates human capital, x_t, from working. If:
  - $d_{1t} = 1$ then $x_{t+1} = x_t$.
  - $d_{2t} = 1$ and $t > 1$ then $x_{t+1} = x_t + 1$.
  - $d_{j=2,t=1} = 1$ then

$$x_2 = \begin{cases} 2 \text{ with probability } 0.5 \\ 1 \text{ with probability } 0.5 \end{cases}$$

- Summarizing, human capital only increases with work, by a unit, except in the first period, when it might jump to two.

Simple Examples of Finite Dependence

Establishing finite dependence in the labor supply example

- When $t > 1$, work one period out of the next two, and:
  - set $\omega_{1,t+1}(t, x_\tau, 2) = 1$, implying $\omega_{2,t+1}(t, x_\tau, 2) = 0$
  - set $\omega_{2,t+1}(t, x_\tau, 1) = 1$, implying $\omega_{1,t+1}(t, x_\tau, 1) = 1$
  - to attain 1-period dependence with $x_{t+2} = x_t + 1$.
- When $t = 1$ after:
  - staying home at $t = 1$ (that is $d_{11} = 1$), work for the next two periods; equivalently set $\omega_{k\tau}(t, x, j)$ so that:

$$\omega_{2,2}(1, 0, 1) = \omega_{2,3}(1, 1, 1) = 1$$

  - working at $t = 1$ (that is $d_{21} = 1$), work in period 2 only if human capital increases one unit at $t = 1$; equivalently set $\omega_{k\tau}(t, x, j)$ so that:

$$\omega_{1,2}(1, 2, 2) = \omega_{2,2}(1, 1, 2) = \omega_{1,3}(1, 2, 2) = 1$$

  - to attain 2-period dependence with $x_3 = 2$.

Simple Examples of Finite Dependence

Another way of establishing finite dependence in the labor supply example

- Alternatively work in the first period ($d_{21} = 1$) and stay home for the next two periods ($d_{12} = d_{13} = 1$); equivalently set $\omega_{k\tau}(t, x, j)$ so that for $x \in \{1, 2\}$:

$$\omega_{1,2}(1, x, 2) = \omega_{1,3}(1, x, 2) = 1$$

- Compare that with staying home in the first period ($d_{11} = 1$), working next period ($d_{22} = 1$), and with probability one half working in period 3; equivalently set $\omega_{k\tau}(t, x, j)$ so that for $x \in \{1, 2\}$:

$$\begin{aligned} \omega_{2,2}(1, 0, 1) &= 1 \\ \omega_{2,3}(1, 1, 1) &= 0.5 \end{aligned}$$

- In both cases the exante distribution of human capital is the same with $\kappa_4(x_4 | t, x_1, j)$ satisfying:

$$\kappa_4(x_4 | 1, 0, 1) = \kappa_4(x_4 | 1, 0, 2) = \begin{cases} 1/2 & \text{if } x_4 = 1 \\ 1/2 & \text{if } x_4 = 2 \end{cases}$$

Simple Examples of Finite Dependence

Nonstationary search model

- Consider a simple search model in which all jobs are temporary, lasting only one period.
- Each period $t \in \{1, \ldots, T\}$ an individual may:
  - stay home by setting $d_{1t} = 1$
  - or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t, time varying job offer arrival rates.
- Experience $x \in \{1, \ldots, X\}$ increases by one unit with each period of work, up to X, and does not depreciate.
- Current utility $u_{jt}(x_t)$ depends on choices, time and experience.

- For all (t, x_t) with $x_t < X$ set:
  - $d_{1t} = 1$ (stay home) and then "apply for employment" with weight:

$$\begin{aligned} \lambda_t / \lambda_{t+1} &= \omega_{k=2,t+1}(t, x_t, i = 1) \\ &= 1 - \omega_{k=1,t+1}(t, x_t, i = 1) \end{aligned}$$

  - $d_{2t} = 1$ (seek work) and then stay home:

$$\omega_{k=1,t+1}\left(t, x_t, j = 2\right) = \omega_{k=1,t+1}\left(t, x_t + 1, j = 2\right) = 1$$

  - to attain one-period dependence since:

$$\kappa_3(x_{t+3}|t, x_t, 1) = \kappa_3(x_{t+3}|t, x_t, 2) = \begin{cases} 1 - \lambda_t \text{ for } x_{t+3} = x_t \\ \lambda_t \text{ for } x_{t+3} = x_t + 1 \end{cases}$$

- Note that if $\lambda_t > \lambda_{t+1}$ then $\omega_{2,t+1}(t, x_t, 1) > 1$ and $\omega_{1,t+1}(t, x_t, 1) = 1 - \lambda_t / \lambda_{t+1} < 0$.

- Suppose the data comprise N observations of the state variables and decisions denoted by $\left\{d_{nt_n}, x_{nt_n}, x_{n,t_n+1}\right\}_{n=1}^{N}$ sampled within a time frame of $t \in \{1, \ldots, S\}$.
- For expositional simplicity suppose the probability of sampling each $x \in \{1, \ldots, X\}$ in $t \in \{1, \ldots, S\}$ is strictly positive.
- M separate instances of finite dependence within that time frame
- Say each pair of choices includes the first choice.
- Label the M paths by $\left(j_m, x_m, t_m, \rho_m\right)$ for $m \in \{1, \ldots, M\}$.
- Assume:
  - $g\left(\epsilon_t\right)$ is known.
  - $\theta \equiv \left(\theta_1, \ldots, \theta_K\right) \in \Theta$, a closed convex set in $\mathbb{R}^K$.
  - $u_{jt}(x) \equiv \widetilde{u}_{jt}(x, \theta)$, where $\widetilde{u}_{jt}(x, \theta)$ is known function.
  - the M instances of finite dependence suffice.

Estimation

The reduced form for a minimum distance (MD) estimator

- For all $(t, x, j) \in \{1, \ldots, S\} \times \{1, \ldots, X\}\, t \in \{1, \ldots, J\}$:
  - define

$$\widehat{p}_{jt}(x) \equiv \frac{\sum_{n=1}^{N} 1\left\{d_{nt_n j} = 1\right\} 1\left\{t_n = t\right\} 1\left\{x_{nt_n} = x\right\}}{\sum_{n=1}^{N} 1\left\{t_n = t\right\} 1\left\{x_{nt_n} = x\right\}}$$

  - estimate the $XJ\mathcal{T}$ CCP vector $p \equiv (p_{11}(1), \ldots, p_{JS}(X))'$ with $\widehat{p}$ formed from $\widehat{p}_{jt}(x)$.
  - Also estimate $f_{jt}(x)$ with $\widehat{f}_{jt}(x)$ in this first stage, for example with a cell estimator (similar to the CCP estimator).

- Define $y(p, f) \equiv (y_1(p, f), \ldots, y_M(p, f))'$ where:

$$y_m(p, f) \equiv \psi_1[p_{t_m}(x_m)] - \psi_{j_m}[p_{t_m}(x_m)]$$
$$+ \sum_{\tau=t_m+1}^{t_m+\rho_m} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta^{\tau-t_m} \psi_k[p_\tau(x_\tau)] \begin{bmatrix} \omega_{k\tau}(t_m, x_\tau, 1)\kappa_\tau(x_\tau | t_m, x_m, 1) - \\ \omega_{k\tau}(t_m, x_\tau, j_m)\kappa_\tau(t_m, x_\tau | x_m, j_m) \end{bmatrix}$$

and $Z(p, f, \theta) \equiv (Z_1(p, f, \theta), \ldots, Z_M(p, f, \theta))'$ where:

$$Z_m(p, f, \theta) \equiv \widetilde{u}_{j_m, t_m}(x_m, \theta)$$
$$- \sum_{\tau=t_m+1}^{t_m+\rho_m} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta_{k\tau}^{\tau-t_m} \widetilde{u}_{k\tau}(x_\tau, \theta) \begin{bmatrix} \omega_{k\tau}(t_m, x_\tau, 1)\kappa_\tau(x_\tau | t_m, x_m, 1) - \\ \omega_{k\tau}(t_m, x_\tau, j_m)\kappa_\tau(x_\tau | t_m, x_m, j_m) \end{bmatrix}$$

- For any M dimensional positive definite matrix W define:

$$\widehat{\theta} \equiv \arg\min_{\theta} \left[y\left(\widehat{p}, \widehat{f}\right) - Z\left(\widehat{p}, \widehat{f}, \theta\right) \right]' W \left[y(\widehat{p}) - Z\left(\widehat{p}, \widehat{f}, \theta\right) \right] \quad (9)$$

- If (θ_0, f) induces p, then $y(p, f) = Z(p, f, \theta_0)$.
- Hence $\widehat{\theta}$ is $\sqrt{N}$ consistent and asymptotically normal.
- Suppose $W = \widehat{W}$, a consistent estimate of the inverse of the asymptotic covariance matrix of $\left(\widehat{p}', \widehat{f}'\right)'$.
- In this case the asymptotic covariance matrix of $\widehat{\theta}$ is:

$$\left[Z\left(\widehat{p}, \widehat{f}, \theta\right) / \partial\theta' \, \widehat{W} Z\left(\widehat{p}, \widehat{f}, \theta\right) / \partial\theta\right]^{-1}$$

- If W is diagonal, (9) reduces to nonlinear least squares (NLS).
- Suppose $\widetilde{u}_{jt}(x, \theta)$ is linear in θ and β is known. If $W = \widehat{W}$ then:

$$\widehat{\theta} = \left\{\frac{\partial Z\left(\widehat{p}, \widehat{f}, \theta\right)'}{\partial\theta} \widehat{W}\left[\partial Z\left(\widehat{p}, \widehat{f}, \theta\right) / \partial\theta\right]\right\}^{-1} \frac{\partial Z\left(\widehat{p}, \widehat{f}, \theta\right)'}{\partial\theta} \widehat{W} y\left(\widehat{p}, \widehat{f}\right)$$

One-Period Dependence in Optimization Problems

Approach

- *Guess and verify* is the only method that has been used to establish finite dependence.
- There is however a systematic way for determining finite period dependence.
- The alternative algorithm iterates between two procedures that checks:
  1. counterfactual outcomes arising from deterministic choices that might either induce or rule out finite dependence.
  2. list the elements of the matrix. The procedure is simpler to establish one-period dependence as there are no intermediate decisions between the initial choice and the choice of weights that generate finite dependence. Hence, checking the rank of a particular matrix is sufficient for determining one-period dependence.
- Much of the intuition for this algorithm can be conveyed by analyzing one period dependence where there are two choices.

One-Period Dependence in Optimization Problems

Equations determining one period dependence in a model with two choices

- Noting that when there are only 2 choices $\kappa_{t+1}(x_{t+1}|t, x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$, reducing one-period dependence to:

$$\begin{aligned} & \kappa_{t+2}(x_{t+2}|t, x_t, 2) \\ \equiv & \sum_{x_{t+1}=1}^{X} \sum_{k=1}^{J=2} \omega_{k.t+1}(t, x_{t+1}, 2) f_{k,t+1}(x_{t+2}|x_{t+1}) f_{2t}(x_{t+1}|x_t) \\ = & \sum_{x_{t+1}=1}^{X} \sum_{k=1}^{J=2} \omega_{k.t+1}(t, x_{t+1}, 1) f_{k,t+1}(x_{t+2}|x_{t+1}) f_{1t}(x_{t+1}|x_t) \\ \equiv & \kappa_{t+2}(x_{t+2}|t, x_t, 1) \end{aligned} \tag{10}$$

- The weights sum to one, implying:

$$\omega_{1.t+1}(t, x_{t+1}, 1) = 1 - \omega_{2.t+1}(t, x_{t+1}, 1)$$

and $\kappa_{t+2}(x_{t+2}|t, x_t, 2)$ is a probability, so:

$$\kappa_{t+2}(X|t, x_t, j) = 1 - \sum_{x_{t+2}=1}^{X-1} \kappa_{t+2}(x_{t+2}|t, x_t, 2)$$

One-Period Dependence in Optimization Problems

Counting the potential equations and unknowns

- Substituting out $\omega_{1.t+1}(t, x_{t+1}, 1)$ we rewrite (10) as:

$$\sum_{x_{t+1}=1}^{X} \left\{ \begin{array}{c} [f_{2,t+1}(x_{t+2}|x_{t+1}) - f_{1,t+1}(x_{t+2}|x_{t+1})] \\ \times \left[\begin{array}{c} \omega_{2.t+1}(t, x_{t+1}, 2) f_{2t}(x_{t+1}|x_t) \\ -\omega_{2.t+1}(t, x_{t+1}, 1) f_{1t}(x_{t+1}|x_t) \end{array} \right] \end{array} \right\}$$

$$= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) [f_{1t}(x_{t+1}|x_t) - f_{2t}(x_{t+1}|x_t)] \quad (11)$$

for all $x_{t+2} \in \{1, 2, \ldots, X-1\}$.

- Nominally this is a linear system:
  - comprising $X-1$ equations
  - in $2X$ weights, $\omega_{2.t+1}(t, 1, k), \ldots, \omega_{2.t+1}(t, X, k)$ for $k \in \{1, 2\}$.
  - with $X-1$ more unknowns than equations.
- Therefore one-period finite dependence holds if and only if a rank condition for the system is satisfied.

- However this is only an upper bound on the number of weights and equations that need to be considered. Define:

$$\begin{aligned} \mathcal{A}_{j,t+1}(x_t) &\equiv \{x \in \mathbb{X} : f_{jt}(x|x_t) > 0\} \\ \mathcal{A}_{t+2}(x_t) &\equiv \left\{ \begin{array}{l} x \in \mathbb{X} : f_{k,t+1}(x|x') > 0 \\ \text{for } x' \in \mathcal{A}_{1,t+1} \bigcup \mathcal{A}_{2,t+1} \text{ and } k \in \{1,2\} \end{array} \right\} \end{aligned}$$

- If $x \notin \mathcal{A}_{j,t+1}(x_t)$ then $\omega_{k.t+1}(t,x,j)$ is irrelevant

$\Longrightarrow$ reducing the number of weights useful that can solve the system.

- If $x \notin \mathcal{A}_{t+2}(x_t)$ then $\kappa_{t+2}(x|t,x_t,2) = \kappa_{t+2}(x|t,x_t,2) = 0$

$\Longrightarrow$ reducing the number of equations to be satisfied.

- Thus the system (11) reduces to $A_{t+2} - 1$ equations linear in $A_{1,t+1} + A_{2,t+1}$ weights, where:

$$A_{j,t+1} \equiv \sum_{x=1}^{X} \mathbf{1}\{x \in \mathcal{A}_{j,t+1}(x_t)\} \qquad A_{t+2} \equiv \sum_{x=1}^{X} \mathbf{1}\{x \in \mathcal{A}_{t+2}(x_t)\}$$

- We can incorporate these two features into the system of equations given by (11). Denote:
  - the $A_{j,t+1}$ dimensional vector of nonzero probabilities in the string $f_{jt}(1|x_t), \ldots, f_{jt}(X|x_t)$ by
    $$\mathsf{K}_{j,t+1}(\mathcal{A}_{j,t+1})$$
  - the $\mathcal{A}_{j,t+1}$ by $\mathcal{A}_{t+2} - 1$ matrix, the first $A_{t+2} - 1$ columns of the $A_{j,t+1} \times A_{t+2}$ transition matrix from $\mathcal{A}_{j,t+1}$ to $\mathcal{A}_{t+2}$ when choice k is made in period $t+1$ (containing elements $f_{k,t+1}(x'|x)$, where $x \in \mathcal{A}_{j,t+1}$ and $x' \in \mathcal{A}_{t+2}$) by
    $$\mathsf{F}_{k,t+1}(\mathcal{A}_{j,t+1})$$
  - the $A_{j,t+1}$ dimensional vector of weights on each of the attainable states, given initial choice j at $t+1$ for setting $d_{2t} = 1$, comprising elements $\omega_{t+1}(x,j)$ for each $x \in \mathcal{A}_{j,t+1}$ by
    $$\Omega_{t+1}\left(\mathcal{A}_{j,t+1}, j\right)$$

One-Period Dependence in Optimization Problems

Rank condition

- Also let:

$$\mathcal{K}_{t+1} \equiv \begin{bmatrix} \mathsf{F}_{1,t+1}(\mathcal{A}_{1,t+1}) \\ -\mathsf{F}_{1,t+1}(\mathcal{A}_{2,t+1}) \end{bmatrix}' \begin{bmatrix} \mathsf{K}_{1,t+1}(\mathcal{A}_{1,t+1}) \\ \mathsf{K}_{2,t+1}(\mathcal{A}_{2,t+1}) \end{bmatrix}$$

$$\mathsf{H}_{t+1} \equiv \begin{bmatrix} \mathsf{F}_{2,t+1}(\mathcal{A}_{2,t+1}) - \mathsf{F}_{1,t+1}(\mathcal{A}_{2,t+1}) \\ \mathsf{F}_{1,t+1}(\mathcal{A}_{1,t+1}) - \mathsf{F}_{2,t+1}(\mathcal{A}_{1,t+1}) \end{bmatrix}$$

- Substituting these transformations into (11), one period dependence holds if and only if there exists an $(A_{1,t+1} + A_{2,t+1})$ vector of unknowns denoted by D_{t+1} solving:

$$\mathcal{K}_{t+1} = \mathsf{H}_{t+1} \begin{bmatrix} \Omega_{t+1}(\mathcal{A}_{j,t+1}, 2) \circ \mathsf{K}_{2,t+1}(\mathcal{A}_{2,t+1}) \\ \Omega_{t+1}(\mathcal{A}_{j,t+1}, 1) \circ \mathsf{K}_{1,t+1}(\mathcal{A}_{1,t+1}) \end{bmatrix} \equiv \mathsf{H}_{t+1}\mathsf{D}_{t+1} \tag{12}$$

  where $\circ$ means element-by-element multiplication.
- A solution to (12) for D_{t+1} exists if and only if the rank of H_{t+1} equals the rank of $\mathsf{H}^*_{t+1} \equiv \left[\mathcal{K}_{t+1} \vdots \mathsf{H}_{t+1}\right]$.