Identification

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Dynamic Optimization with Conditional Independence

Discrete time with finite choice sets and a finite state space

- Dropping the superscript * notation, suppose that each period $t \in \{1, \ldots, T\}$, the agent observes the realization (x_t, ϵ_t), and chooses d_t to sequentially maximize:

$$E\left\{\sum_{\tau=t}^{T} \sum_{j=1}^{J} \beta^{\tau-1} d_{j\tau}\left[u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right] | x_t, \epsilon_t\right\} \quad (1)$$

  where:

  - An integer $T \leq \infty$ denotes the horizon of the optimization problem.
  - the individual chooses amongst J mutually exclusive actions.
  - $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ where $d_{jt} = 1$ if action $j \in \{1, \ldots, J\}$ is taken at time t and $d_{jt} = 0$ if action j is not taken at t.
  - $x_t \in \{1, \ldots, X\}$ for some finite positive integer X for each t.
  - $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ where $\epsilon_{jt} \in \mathbb{R}$ for all (j, t).
  - conditional independence holds, meaning:

$$g_{t,j,x,\epsilon}\left(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t\right) = g_{t+1}\left(\epsilon_{t+1} | x_{t+1}\right) f_{jt}(x_{t+1} | x_t)$$

- Denote the optimal decision rule at t as $d_t^o(x_t, \epsilon_t)$, with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$ and define the *social surplus function* as:

$$V_t(x_t) \equiv E\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t-1}d_{j\tau}^{o}\left(x_\tau, \epsilon_\tau\right)\left(u_{j\tau}(x_\tau)+\epsilon_{j\tau}\right)\right\}$$

- The *conditional value function*, $v_{jt}(x_t)$, is defined as:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta\sum_{x=1}^{X}V_{t+1}(x)f_{jt}\left(x|x_t\right)$$

- Integrating $d_{jt}^o(x_t, \epsilon)$ over $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_J)$ the CCPs are defined by:

$$p_{jt}(x_t) \equiv E\left[d_{jt}^{o}\left(x_t, \epsilon\right)|x_t\right] = \int d_{jt}^{o}\left(x_t, \epsilon\right)g_t\left(\epsilon\,|x_t\right)d\epsilon$$

- The starting point for our analysis is to define differences in the conditional valuation functions as:

$$\Delta v_{jkt}(x) \equiv v_{jt}(x) - v_{kt}(x)$$

- Although there are $J(J-1)$ differences all but $(J-1)$ are linear combinations of the $(J-1)$ basis functions.
- For example setting the basis functions as:

$$\Delta v_{jt}(x) \equiv v_{jt}(x) - v_{Jt}(x)$$

  then clearly:

$$\Delta v_{jkt}(x) = \Delta v_{jt}(x) - \Delta v_{kt}(x)$$

- Without loss of generality we focus on this particular basis function.

Inversion

Each CCP is a mapping of differences in the conditional valuation functions

- Using the definition of $\Delta v_{jt}(x)$:

$$
\begin{aligned}
p_{jt}(x) &\equiv \int d_{jt}^{o}(x,\epsilon)\, g_t(\epsilon\,|x)\, d\epsilon \\
&= \int I\left\{\epsilon_k \leq \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall\ k \neq j\right\} g_t(\epsilon\,|x)\, d\epsilon \\
&= \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{1t}(x)} \cdots \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{J-1,t}(x)} \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)} g_t(\epsilon\,|x)\, d\epsilon
\end{aligned}
$$

- Noting $g_t(\epsilon\,|x) \equiv \partial^J G_t(\epsilon\,|x) / \partial\epsilon_1, \ldots, \partial\epsilon_J$, integrate over $(\epsilon_1, \ldots, , \epsilon_{j-1}, \epsilon_{j+1} \ldots, \epsilon_J)$.
- Denoting $G_{jt}(\epsilon\,|x) \equiv \partial G_t(\epsilon\,|x) / \partial\epsilon_j$, yields:

$$
p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \ldots \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} \,|x \right) d\epsilon_j
$$

Inversion

There are as many CCPs as there are conditional valuation functions

- For any vector $J-1$ dimensional vector $\delta \equiv (\delta_1, \ldots, \delta_{J-1})$ define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}(\epsilon_j + \delta_j - \delta_1, \ldots, \epsilon_j, \ldots, \epsilon_j + \delta_j \,|x) \, d\epsilon_j$$

- We interpret $Q_{jt}(\delta, x)$ as the probability taking action j in a static random utility model (RUM) where the payoffs are $\delta_j + \epsilon_j$ and the probability distribution of disturbances is given by $G_t(\epsilon \,|x)$.
- It follows from the definition of $Q_{jt}(\delta, x)$ that:

$$0 \leq Q_{jt}(\delta, x) \leq 1 \text{ for all } (j, t, \delta, x) \text{ and } \sum_{j=1}^{J-1} Q_{jt}(\delta, x) \leq 1$$

- In particular the previous slide implies that for any given (j, t, x):

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x \right) d\epsilon_j \equiv Q_{jt}(\Delta v_t(x), x)$$

Inversion

Proposition 1 of Hotz and Miller (1993)

Theorem (Inversion)

For each (t, δ, x) *define:*

$$Q_t(\delta, x) \equiv (Q_{1t}(\delta, x), \ldots Q_{J-1,t}(\delta, x))'$$

Then the vector function $Q_t(\delta, x)$ *is invertible in* δ *for each* (t, x)*.*

- Note that $p_{Jt}(x) = Q_{Jt}(\Delta v_t, x)$ is a linear combination of the other equations in the system because $\sum_{k=1}^{J} p_k = 1$.
- Let $p \equiv (p_1, \ldots, p_{J-1})$ where $0 \leq p_j \leq 1$ for all $j \in \{1, \ldots, J-1\}$ and $\sum_{j=1}^{J-1} p_j \leq 1$. Denote the inverse of $Q_{jt}(\Delta v_t, x)$ by $Q_{jt}^{-1}(p, x)$.
- The inversion theorem implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}[p_t(x), x] \\ \vdots \\ Q_{J-1,t}^{-1}[p_t(x), x] \end{bmatrix}$$

Inversion

Using the inversion theorem

- In finessing optimization and integration by exploiting conditional independence, how far can the three applications described in the previous two lectures be extended?
- We use the Inversion Theorem to:
  1. provide empirically tractable representations of the conditional value functions.
  2. analyze identification in dynamic discrete choice models.
  3. provide convenient parametric forms for the density of ϵ_t that generalize the Type 1 Extreme Value distribution.
  4. generalize the renewal and terminal state properties exploited in the first two examples, by obtaining restrictions on the state variable transitions used to implement CCP estimators.
  5. introduce new methods for incorporating unobserved state variables.

Corollaries of the Inversion Theorem

Identifying the policy function

- From the definition of the optimal decision rule, and then appealing to the inversion theorem:

$$
\begin{aligned}
d_{jt}^{o}\left(x_t,\epsilon_t\right) &= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt}-\epsilon_{jt}\leq v_{jt}(x)-v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt}-\epsilon_{jt}\leq \begin{array}{l} v_{jt}(x)-v_{Jt}(x_t) \\ -\left[v_{kt}(x)-v_{Jt}(x_t)\right] \end{array}\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt}-\epsilon_{jt}\leq \Delta v_{jt}(x)-\Delta v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt}-\epsilon_{jt}\leq Q_{jt}^{-1}\left[p_t(x),x\right]-Q_{kt}^{-1}\left[p_t(x),x\right]\right\}
\end{aligned}
$$

- If $G_t\left(\epsilon\,|x\right)$ is known and the data generating process (DGP$)$ is (x_t, d_t), then $p_t(x)$ and hence $d_t^{o}\left(x_t,\epsilon_t\right)$ are identified.

Corollaries of the Inversion Theorem

Definition of the conditional value function correction

- Define the conditional value function correction as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- In stationary settings, we drop the t subscript and write:

$$\psi_j(x) \equiv V(x) - v_j(x)$$

- Suppose that instead of taking the optimal action she committed to taking action j instead. Then the expected lifetime utility would be:

$$v_{jt}(x_t) + E_t[\epsilon_{jt} | x_t]$$

so committing to j before ϵ_t is revealed entails a loss of:

$$V_t(x_t) - v_{jt}(x_t) - E_t[\epsilon_{jt} | x_t] = \psi_{jt}(x) - E_t[\epsilon_{jt} | x_t]$$

- For example if $E_t[\epsilon_t | x_t] = 0$, the loss simplifies to $\psi_{jt}(x)$.

Corollaries of the Inversion Theorem

Identifying the conditional value function correction

- From their respective definitions:

$$
\begin{aligned}
& V_t(x) - v_{it}(x) \\
= \quad & \sum_{j=1}^{J} \left\{ p_{jt}(x) \left[v_{jt}(x) - v_{it}(x) \right] + \int \epsilon_{jt} d_{jt}^{o} \left(x_t, \epsilon_t \right) g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t \right\}
\end{aligned}
$$

- But:

$$
v_{jt}(x) - v_{it}(x) = Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{it}^{-1} \left[p_t(x), x \right]
$$

and

$$
\begin{aligned}
& \int \epsilon_{jt} d_{jt}^{o} \left(x, \epsilon_t \right) g \left(\epsilon_t \left| x \right. \right) d\epsilon_t \\
= \quad & \int \prod_{k=1}^{J} 1 \left\{ \begin{array}{l} \epsilon_{kt} - \epsilon_{jt} \\ \leq Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{kt}^{-1} \left[p_t(x), x \right] \end{array} \right\} \epsilon_{jt} g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t
\end{aligned}
$$

- Therefore $\psi_{it}(x) \equiv V_t(x) - v_{it}(x)$ is identified if $G_t \left(\epsilon \left| x \right. \right)$ is known and $\left(x_t, d_t \right)$ is the DGP.

Conditional Valuation Function Representation

Telescoping one period forward

- From its definition:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}(x_{t+1}|x_t)$$

- Substituting for $V_{t+1}(x_{t+1})$ using conditional value function correction we obtain for any k:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} \left[v_{k,t+1}(x) + \psi_{k,t+1}(x) \right] f_{jt}(x|x_t)$$

- We could repeat this procedure ad infinitum, substituting in for $v_{k,t+1}(x)$ by using the definition for $\psi_{kt}(x)$.

Conditional Valuation Function Representation

Recursively defining the distribution of future state variables

- To formalize this idea, consider a random sequence of weights from t to T which begins with $\omega_{jt}(x_t, j) = 1$.
- For periods $\tau \in \{t+1, \ldots, T\}$, the choice sequence maps x_τ and the initial choice j into

$$\omega_\tau(x_\tau, j) \equiv \{\omega_{1\tau}(x_\tau, j), \ldots, \omega_{J\tau}(x_\tau, j)\}$$

where $\omega_{k\tau}(x_\tau, j)$ may be negative or exceed one but:

$$\sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) = 1$$

.

- The weight of state $x_{\tau+1}$ conditional on following the choices in the sequence is recursively defined by $\kappa_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and for $\tau = t+1, \ldots, T$:

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j)\, f_{k\tau}(x_{\tau+1}|x_\tau)\kappa_{\tau-1}(x_\tau|x_t, j)$$

Theorem (Representation)

For any state $x_t \in \{1, \ldots, X\}$, choice $j \in \{1, \ldots, J\}$ and weights $\omega_\tau(x_\tau, j)$ defined for periods $\tau \in \{t, \ldots, T\}$:

$$\begin{aligned} v_{jt}(x_t) &= u_{jt}(x_t) \\ &+ \sum_{\tau=t+1}^{T} \sum_{k=1}^{J} \sum_{x=1}^{X} \beta^{\tau-t} \left[u_{k\tau}(x) + \psi_k[p_\tau(x)]\right] \omega_{k\tau}(x,j) \kappa_{\tau-1}(x|x_t, j) \end{aligned}$$

- The theorem yields an alternative expression for $v_{jt}(x_t)$ that dispenses with recursive maximization.
- Intuitively, the individuals have already solved their optimization problem, so their decisions, as reflected in their CCPs, are informative of their value functions.

- Are there tractable distributions $G_t(\epsilon | x)$ aside from the Type 1 Extreme Value?
- To keep the approach operational we have to compute $\psi_k(p)$ for at least some k.
- Suppose ϵ is drawn from the GEV distribution function:

$$G(\epsilon_1, \epsilon_2, \ldots, \epsilon_J) \equiv \exp\left[-\mathcal{H}(\exp[-\epsilon_1], \exp[-\epsilon_2], \ldots, \exp[-\epsilon_J])\right]$$

where $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ satisfies the following properties:

1. $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ is nonnegative, real valued, and homogeneous of degree one;
2. $\lim \mathcal{H}(Y_1, Y_2, \ldots, Y_J) \to \infty$ as $Y_j \to \infty$ for all $j \in \{1, \ldots, J\}$;
3. for any distinct $(i_1, i_2, \ldots, i_r)$ the cross derivative $\partial\mathcal{H}(Y_1, Y_2, \ldots, Y_J) / \partial Y_{i_1}, Y_{i_2}, \ldots, Y_{i_r}$ is nonnegative for r odd and nonpositive for r even.

- Suppose $G(\epsilon)$ factors into two independent distributions, one a nested logit, and the other any GEV distribution.
- Let $\mathcal{J}$ denote the set of choices in the nest and denote the other distribution by $G_0(Y_1, Y_2, \ldots, Y_K)$ let K denote the number of choices that are outside the nest.
- Then:

$$G(\epsilon) \equiv G_0(\epsilon_1, \ldots, \epsilon_K) \exp\left[-\left(\sum_{j \in \mathcal{J}} \exp\left[-\epsilon_j/\sigma\right]\right)^{\sigma}\right]$$

- The correlation of the errors within the nest is given by $\sigma \in [0, 1]$ and errors within the nest are uncorrelated with errors outside the nest. When $\sigma = 1$, the errors are uncorrelated within the nest, and when $\sigma = 0$ they are perfectly correlated.

Generalized Extreme Values

Lemma 2 of Arcidiacono and Miller (2011)

- Define $\phi_j(Y)$ as a mapping into the unit interval where

$$\phi_j(Y) = Y_j \mathcal{H}_j \left(Y_1, \ldots, Y_J \right) / \mathcal{H} \left(Y_1, \ldots, Y_J \right)$$

- Since $\mathcal{H}_j \left(Y_1, \ldots, Y_J \right)$ and $\mathcal{H} \left(Y_1, \ldots, Y_J \right)$ are homogeneous of degree zero and one respectively, $\phi_j(Y)$ is a probability, because $\phi_j(Y) \geq 0$ and $\sum_{j=1}^{J} \phi_j(Y) = 1$.

Lemma (GEV correction factor)

When ϵ_t is drawn from a GEV distribution, the inverse function of $\phi(Y) \equiv (\phi_2(Y), \ldots \phi_J(Y))$ exists, which we now denote by $\phi^{-1}(p)$, and:

$$\psi_j(p) = \ln \mathcal{H} \left[1, \phi_2^{-1}(p), \ldots, \phi_J^{-1}(p) \right] - \ln \phi_j^{-1}(p) + \gamma$$

Lemma

For the nested logit $G(\epsilon_t)$ defined above:

$$\psi_j(p) = \gamma - \sigma \ln(p_j) - (1-\sigma) \ln\left(\sum_{k \in \mathcal{J}} p_k\right)$$

- Note that $\psi_j(p)$ only depends on the conditional choice probabilities for choices that are in the nest: the expression is the same no matter how many choices are outside the nest or how those choices are correlated.
- Hence, $\psi_j(p)$ will only depend on $p_{j'}$ if ϵ_{jt} and $\epsilon_{j't}$ are correlated. When $\sigma = 1$, ϵ_{jt} is independent of all other errors and $\psi_j(p)$ only depends on p_j.

- The optimization model is fully characterized by the time horizon, the utility flows, the discount factor, the transition matrix of the observed state variables, and the distribution of the unobserved variables, summarized with the notation (T, β, f, g, u).
- The data comprise observations for a real or synthetic panel on the observed part of the state variable, x_t, and decision outcomes, d_t.
- In our analysis, let $S \leq T$ denote the last date for which data is available (for a real or synthetic cohort).
- Following most of the empirical work in this area we consider identification when (T, β, f, g) are assumed to be known.
- Thus the goal is to identify u from (x_t, d_t) when (T, β, f, g) is known.

Identifying the Primitives

Observational Equivalence

- It is widely believed that u is only identified relative to one choice per period for each state.
- Can we say more than that?
- For each (x, t) let $l(x, t) \in \{1, \ldots, J\}$ denote any arbitrarily defined normalizing action and $c_t(x) \in \mathbb{R}$ its associated benchmark flow utility, meaning $u^*_{l(x,t),t}(x) \equiv c_t(x)$.
- Assume $\{c_t(x)\}_{t=1}^{T}$ is bounded for each $x \in \{1, \ldots, X\}$.
- Let $\kappa^*_\tau(x_{\tau+1}|x_t, j)$ denote the probability distribution of $x_{\tau+1}$, given a state of x_t taking action j at t, and then repeatedly taking the normalized action from period $t+1$ through to period τ.
- Thus $\kappa^*_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and for $\tau \in \{t+1, \ldots, T\}$:

$$\kappa^*_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x=1}^{X} f_{l(x,\tau),\tau}(x_{\tau+1}|x)\kappa^*_{\tau-1}(x|x_t, j) \qquad (2)$$

Theorem (Observational Equivalence, Arcidiacono and Miller, 2020)

For each $R \in \{1, 2, \ldots\}$*, define for all* $x \in \{1, \ldots, X\}$, $j \in \{1, \ldots, J\}$ *and* $t \in \{1, \ldots, R\}$*:*

$$u^*_{jR}(x) \equiv u_{jR}(x) + c_R(x) - u_{l(x,R),R}(x) \tag{3}$$

$$\begin{aligned} u^*_{jt}(x) \equiv\ & u_{jt}(x) + c_t(x) - u_{l(x,t),t}(x) \\ & + \sum_{\tau=t+1}^{R} \sum_{x'=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} \left[c_\tau(x') - u_{l(x,\tau),\tau}(x') \right] \times \\ \left[\kappa^*_{\tau-1}(x'|x_t, l(x,t)) - \kappa^*_{\tau-1}(x'|x_t, j) \right] \end{array} \right\} \end{aligned} \tag{4}$$

(T, β, f, g, u^*)*, is observationally equivalent to* (T, β, f, g, u) *in the limit of* $R \rightarrow T$*. Conversely suppose* (T, β, f, g, u^*) *is observationally equivalent to* (T, β, f, g, u)*. For each date and state select any action* $l(x,t) \in \{1, \ldots, J\}$ *with payoff* $u^*_{l(x,t),t}(x) \equiv c_t(x) \in \mathbb{R}$*. Then* (3) *and* (4) *hold for all* (t, x, j)*.*

Corollary

Suppose $u_{jt}(x) = u_j(x)$ *and let* $u_j \equiv (u_j(1), \ldots, u_j(X))'$. *Similarly suppose* $f_{jt}(x_{t+1}|x_t) = f_j(x_{t+1}|x_t)$ *for all* $t \in \{1, 2, \ldots\}$. *Denote by* $I(x)$ *the normalizing action for that state, with true payoff vector* $u_I = \left(u_{I(1)}(1), \ldots, u_{I(X)}(X)\right)'$, *and assume* $c(x) \equiv (c(1), \ldots, c(X))'$ *is bounded for each* $x \in \{1, 2, \ldots\}$. *Then* (4) *reduces to:*

$$u_j^* = u_j + [I - \beta F_j][I - \beta F_I]^{-1}(c - u_I) \tag{5}$$

where $u_j^* \equiv \left(u_j^*(1), \ldots, u_j^*(X)\right)'$, *the* X *dimensional identity matrix is denoted by* I, *and:*

$$F_j \equiv \begin{bmatrix} f_j(1|1) & \ldots & f_j(X|1) \\ \vdots & \ddots & \vdots \\ f_j(1|X) & \ldots & f_j(X|X) \end{bmatrix}, \quad F_I \equiv \begin{bmatrix} f_{I(1)}(1|1) & \ldots & f_{I(1)}(X|1) \\ \vdots & \ddots & \vdots \\ f_{I(X)}(1|X) & \ldots & f_{I(X)}(X|X) \end{bmatrix}$$

- A common normalization is to let $I(x, \tau) = 1$ and $c_t(x) = 0$ for all (t, x), normalizing the payoff from the first choice to zero by defining $u_{1t}^*(x) \equiv 0$, and interpreting the payoffs for other actions as net of, or relative to, the current payoff for the first choice.
- The theorem shows that with the important exception of the static model (when $T = 1$), this interpretation is misleading.
- Define $\kappa_\tau(x_{\tau+1}|x_t, j)$ by setting $f_{I(x,\tau),\tau}(x_{\tau+1}|x) = f_{1\tau}(x_{\tau+1}|x)$ in (2), if $T < \infty$ then (3) and (4) simplify to:

$$u_{jT}^*(x) = u_{jT}(x) - u_{1T}(x)$$

and:

$$\begin{aligned} u_{jt}^*(x) &= u_{jt}(x) - u_{1t}(x) \\ &\quad - \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} u_{1\tau}(x_\tau) \left[\kappa_{\tau-1}(x_\tau|x_t, 1) - \kappa_{\tau-1}(x_\tau|x_t, j)\right] \end{aligned}$$

Identifying the Primitives

Identification off long panels (Arcidiacono and Miller,2020)

Theorem (Identification)

For all j, t, and x:

$$u_{jt}(x) = u_{1t}(x) + \psi_{1t}(x) - \psi_{jt}(x) + \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} [u_{1\tau}(x_\tau) + \psi_{1t}(x_\tau)] \times \\ [\kappa_{\tau-1}(x_\tau|x,1) - \kappa_{\tau-1}(x_\tau|x,j)] \end{array} \right\} \quad (6)$$

In stationary models, define $\Psi_j \equiv \left[\psi_j(1) \ldots \psi_j(X)\right]'$, *and for all j:*

$$u_j = \Psi_1 - \Psi_j - u_1 + \beta\left(F_1 - F_j\right)\left[I - \beta F_1\right]^{-1}\left(\Psi_1 + u_1\right) \quad (7)$$

- If (T, β, f, g) is known, and if a payoff, say the first, is also known for every state and time, then u is identified.

Identifying the Primitives

Proving the theorem

- Specialize the mixed decision rule by always taking the first action to obtain:

$$
\begin{aligned}
v_{jt}(x) &= u_{jt}(x) \\
&\quad + \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left[u_{1\tau}(x_\tau) + \psi_{1\tau}(x_\tau) \right] \kappa_{\tau-1}(x_\tau | x, j)
\end{aligned}
$$

- Subtract from the expression above the corresponding expression for $v_{1t}(x_t)$ yielding:

$$
\begin{aligned}
& v_{jt}(x) - u_{jt}(x) - \left[v_{1t}(x) - u_{1t}(x) \right] \\
= \; & \psi_{1t}(x) - \psi_{jt}(x) - u_{jt}(x) + u_{1t}(x) \\
= \; & \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} \left[u_{1\tau}(x_\tau) + \psi_{1t}(x_\tau) \right] \times \\ \left[\kappa_{\tau-1}(x_\tau | x, 1) - \kappa_{\tau-1}(x_\tau | x, j) \right] \end{array} \right\}
\end{aligned}
$$

- The theorem follows from rearrangement.