Finite Dependence

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December 2023

Introduction

Motivation

- Estimation of dynamic discrete choice models is complicated by the calculation of expected future payoffs.
- These complications are mitigated when finite dependence holds.
- *Intuitively, ρ period dependence holds when two sequences of weighted (nonoptimal) choices leading off from different initial choices generate the same distribution of state variables $\rho + 1$ periods later.*
- Exploiting the finite dependence property reduces the number of CCPs to estimate.
- Most empirical applications have the finite dependence property.
- Under the conditional independence assumption, finite dependence has empirical content (can be tested) without specifying utilities.
- The finite dependence property depends on the state transition functions but not on optimizing behavior.

Introduction

Framework

- $t \in \{1, \ldots, T\}$ stands for time, where $T \leq \infty$.
- $x_t \in \{1, \ldots, X\} \equiv \mathbb{X}$ is the state at t, where $\mathbb{X}$ is a finite set.
- $j \in \{1, \ldots, J\}$ is a (mutually exclusive) choice.
- $d_{jt} = 1$ if j is picked at t and otherwise $d_{jt} = 0$.
- $f_{jt}(x_{t+1}|x_t)$ is the probability of x_{t+1} occurring in period $t+1$ conditional on x_t and $d_{jt} = 1$.
- $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ has continuous support and is IID over t with PDF $g(\epsilon_t)$ satisfying $E[\max\{\epsilon_{1t}, \ldots, \epsilon_{Jt}\}] \leq \bar{\epsilon} < \infty$.
- For some $\beta \in (0, 1)$, the individual sequentially chooses the vector $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ to maximize:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1}d_{jt}\left[u_{jt}(x_t) + \epsilon_{jt}\right]\right\} \tag{1}$$

- $d_t^o(x_t, \epsilon_t)$ is the optimal decision rule with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$.
- $p_t(x_t) \equiv (p_{1t}(x_t), \ldots, p_{Jt}(x_t))$ are the CCPs, where:

$$p_{jt}(x_t) \equiv \int d_{jt}^o(x_t, \epsilon_t)\, g(\epsilon_t)\, d\epsilon_t \tag{2}$$

- $V_t(x_t)$ is the ex-ante value function defined as :

$$V_t(x_t) \equiv E\left\{ \sum_{\tau=t}^{T} \sum_{j=1}^{J} \beta^{\tau-t} d_{j\tau}^o(x_\tau, \epsilon_\tau)\left(u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right) \right\}$$

- The conditional value function for action j defined as:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x_{t+1}=1}^{X} V_{t+1}(x_{t+1}) f_{jt}(x_{t+1}|x_t) \tag{3}$$

- The conditional value function correction for action j is defined:

$$\psi_j[p_t(x)] \equiv V_t(x) - v_{jt}(x) \tag{4}$$

Introduction

Weighting schemes and the transitions they induce

- We defined a weight sequence $\{\omega_{k\tau}(x_\tau, j, t)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$ satisfying:

$$|\omega_{k\tau}(x_\tau, j, t)| < \infty \text{ and } \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j, t) = 1$$

- To emphasize the potential dependence of the weight sequence on (j, x_t, t) we could have written:

$$\omega_{k\tau}(x_\tau, j, x_t, t) \text{ rather than } \omega_{k\tau}(x_\tau, j, t)$$

- Similarly recall that $\kappa_{t+1}(x_{t+1}|j, x_t, t) \equiv f_{jt}(x_{t+1}|x_t)$ and:

$$\kappa_{\tau+1}(x_{\tau+1}|j, x_t, t) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j, t) f_{k\tau}(x_{\tau+1}|x_\tau) \kappa_\tau(x_\tau|j, x_t, t) \tag{5}$$

- To be more explicit we could have written:

$$\kappa_{\tau+1}^{(\omega)}(x_{\tau+1}|j, x_t, t) \text{ rather than } \kappa_{\tau+1}(x_{\tau+1}|j, x_t, t)$$

Introduction

Conditional value function representation telescoping a finite number of periods

- The representation given in the second lecture implies:

$$v_{jt}(x_t) = V_t(x) - \psi_j[p_t(x_t)] = \tag{6}$$

$$\left\{ \begin{array}{l} u_{jt}(x_t) + \sum_{x=1}^{X} \beta^{\rho+1} V_{t+\rho+1}(x) \kappa_{t+\rho+1}(x|j, x_t, t) \\ + \sum_{\tau=t+1, k=1, x_\tau=1}^{t+\rho, J, X} \beta^{\tau-t} \left[\begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right] \left[\begin{array}{l} \omega_{k\tau}(x_\tau, j, t) \\ \times \kappa_\tau(x_\tau|j, x_t, t) \end{array} \right] \end{array} \right\}$$

- In this formulation:
  - $V_{t+\rho+1}(x)$ captures the value from $t+\rho+1$ having reached x.
  - $\kappa_{t+\rho+1}(x|j, x_t, t) \equiv \kappa^{(\omega)}_{t+\rho+1}(x|j, x_t, t)$ is the probability of reaching x by following $\{\omega_{k\tau}(x_\tau, j, t)\}^{(J,t+\rho)}_{(k,\tau)=(1,t+1)}$ from (t, x_t).
  - $\psi_k[p_\tau(x_\tau)]$ are correction terms for nonoptimal behavior.

Finite Dependence

Definition

- The pair of choices $\{i, j\}$ exhibits *ρ-period dependence* at (t, x_t) if there exist a pair of sequences of decision weights:

$$\{\omega_{k\tau}(x_\tau, i, t)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)} \text{ and } \{\omega_{k\tau}(x_\tau, j, t))\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$$

such that for all $x_{t+\rho+1} \in \{1, \ldots, X\}$:

$$\kappa_{t+\rho+1}(x_{t+\rho+1}|i, x_t, t) = \kappa_{t+\rho+1}(x_{t+\rho+1}|j, x_t, t)$$

- Finite dependence:
  1. trivially holds for $\rho = T - t$ when $T < \infty$, but only merits attention when $\rho < T - t$.
  2. extends to games by conditioning on the player as well.
  3. might hold for some choice pairs but not others, and for certain states but not others.
  4. could be defined for mixed choices to start the sequence, not just deterministic moves; this analysis extends to the more general case.

Finite Dependence

Representing utility

- If there is finite dependence for (i, j, x_t, t), then:

$$u_{jt}(x_t) + \psi_j[p_t(x_t)] - u_{it}(x_t) - \psi_i[p_t(x_t)] =$$

$$\sum_{(k,\tau,x_\tau)=(1,t+1,1)}^{(J,t+\rho,X)} \beta^{\tau-t} \left\{ \begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right\} \left[\begin{array}{l} \omega_{k\tau}(x_\tau, i, t)\kappa_\tau(x_\tau|i, x_t, t) \\ -\omega_{k\tau}(x_\tau, j, t)\kappa_\tau(x_\tau|j, x_t, t) \end{array} \right] \tag{7}$$

- To derive this equation:
  - replace $v_{jt}(x)$ with $V_t(x) - \psi_j[p_t(x_t)]$ in (6)
  - form an analogous equation for i
  - difference the two resulting equations
  - note the $V_t(x)$ and $V_{t+\rho+1}(x)\kappa_{t+\rho+1}(x|x_t, j, t)$ terms cancel.

Simple Examples of Finite Dependence

Terminal choices

- A *terminal choice* ends the optimization problem or the game for that particular player.
- They are quite common in empirical applications.
- Supposing the first choice a terminal choice, then $u_{1t}(x_t)$ can be interpreted as the final payoff at (t, x_t) if that choice is made then.
- Setting $\omega_{k,t+1}(t, x_{t+1}, i) = 0$ for all (x, i) and $k \neq 1$, Equation (7) reduces to:

$$\begin{aligned} & u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = & \sum_{x_{t+1}=1}^{X} \beta \left\{ u_{1,t+1}(x_{t+1}) + \psi_1 \left[p_{t+1}(x_{t+1}) \right] \right\} f_{jt}(x_{t+1}|x_t) \end{aligned}$$

Simple Examples of Finite Dependence

Renewal choices are also quite common in empirical applications

- Similarly a *renewal choice* yields a probability distribution of the state variable next period that does not depend on the current state.
- If the first choice is a renewal choice, then for all $j \in \{1, \ldots, J\}$:

$$\begin{aligned}\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \sum_{x_{t+1}=1}^{X} f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \qquad (8)\end{aligned}$$

- In this case Equation (7) implies:

$$\begin{aligned}& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = & \sum_{x=1}^{X} \beta \{u_{1,t+1}(x) + \psi_1[p_{t+1}(x)]\} [f_{jt}(x|x_t) - f_{1t}(x|x_t)]\end{aligned}$$

Simple Examples of Finite Dependence

An example of 2-period finite dependence

- How does finite dependence work when $\rho > 1$?
- Consider the following model of labor supply and human capital.
- In each of T periods an individual chooses:
  - $d_{2t} = 1$ to work
  - $d_{1t} = 1$ to stay home.
- She accumulates human capital, x_t, from working:
  - If $d_{1t} = 1$ then $x_{t+1} = x_t$.
  - If $d_{2t} = 1$ and $t > 1$ then $x_{t+1} = x_t + 1$.
  - If $d_{j=2,t=1} = 1$ then

$$x_2 = \begin{cases} 2 \text{ with probability } 0.5 \\ 1 \text{ with probability } 0.5 \end{cases}$$

- Summarizing, human capital only increases with work, by a unit, except in the first period, when it might jump to two.

Simple Examples of Finite Dependence

Establishing finite dependence in the labor supply example

- When $t > 1$, one-period dependence can be attained with $x_{t+2} = x_t + 1$ by working one period out of the next two:
  - If $d_{2t} = 1$ set $\omega_{1,t+1}(x_{t+1} = x_t + 1, j = 2, t) = 1$.
  - If $d_{1t} = 1$ set $\omega_{2,t+1}(x_{t+1} = x_t, j = 1, t) = 1$.
- When $t = 1$ two-period dependence can be attained with $x_3 = 2$:
  - Stay home at $t = 1$ (setting $d_{11} = 1$), and then work for two periods:

$$\begin{aligned} 1 &= \omega_{k=2,\tau=2}(x_2 = 0, j = 1, t = 1) \\ &= \omega_{k=2,\tau=3}(x_3 = 1, j = 1, t = 1) \end{aligned}$$

  - Work at $t = 1$ (setting $d_{21} = 1$), and work in $t = 2$ only if human capital increases by one unit at $t = 1$:

$$\begin{aligned} 1 &= \omega_{k=1,\tau=2}(x_2 = 2, j = 2, t = 1) \\ &= \omega_{k=2,\tau=2}(x_2 = 1, j = 2, t = 1) \\ &= \omega_{k=1,\tau=3}(x_2 = 3, j = 2, t = 1) \end{aligned}$$

Simple Examples of Finite Dependence

Another way of establishing finite dependence in the labor supply example

- Alternatively work in the first period ($d_{21} = 1$) and stay home for the next two periods ($d_{12} = d_{13} = 1$), so that for $x \in \{1, 2\}$:

$$\omega_{k=1,\tau=2}(x, j = 2, t = 1) = \omega_{k=1,\tau=3}(x, j = 2, t = 1) = 1$$

- Compare that with staying home in the first period ($d_{11} = 1$), working next period ($d_{22} = 1$), and with probability one half working in period 3, so that for $x \in \{1, 2\}$:

$$\begin{aligned} \omega_{k=2,\tau=2}(x_2 &= 0, j = 1, t = 1) = 1 \\ \omega_{k=2,\tau=3}(x_3 &= 1, j = 1, t = 1) = 0.5 \end{aligned}$$

- In both cases the exante distribution of human capital is:

$$\begin{aligned} \kappa_{\tau=4}(x_4 | j = 1, x_1 = 0, t = 1) &= \kappa_{\tau=4}(x_4 | j = 2, x_1 = 0, t = 1) \\ &= \begin{cases} 1/2 & \text{if } x_4 = 1 \\ 1/2 & \text{if } x_4 = 2 \end{cases} \end{aligned}$$

Simple Examples of Finite Dependence

Nonstationary search model

- Consider a simple search model in which all jobs are temporary, lasting only one period.
- Each period $t \in \{1, \ldots, T\}$ an individual may:
  - stay home by setting $d_{1t} = 1$
  - or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t, time varying job offer arrival rates.
- Experience $x \in \{1, \ldots, X\}$ increases by one unit with each period of work, up to X, and does not depreciate.

Simple Examples of Finite Dependence

Finite dependence in this search model

- For all (x_t, t) with $x_t < X$ set:
  - $d_{1t} = 1$ (stay home) and then "apply for employment" with weight:

$$\begin{aligned} \lambda_t / \lambda_{t+1} &= \omega_{k=2,t+1}(x_t, j = 1, t) \\ &= 1 - \omega_{k=1,t+1}(x_t, j = 1, t) \end{aligned}$$

  - $d_{2t} = 1$ (seek work) and then stay home:

$$\omega_{k=1,t+1}(x_t, j = 2, t) = \omega_{k=1,t+1}(x_t + 1, j = 2, t) = 1$$

- This pair of sequencing attains one-period dependence since:

$$\kappa_3(x_{t+3}|j = 1, x_t, t) = \kappa_3(x_{t+3}|j = 2, x_t, t) = \begin{cases} 1 - \lambda_t \text{ for } x_{t+3} = x_t \\ \lambda_t \text{ for } x_{t+3} = x_t + 1 \end{cases}$$

- Note that if $\lambda_t > \lambda_{t+1}$ then $\omega_{k=2,t+1}(x_t, j = 1, t) > 1$ and:

$$\omega_{k=1,t+1}(x_{t+1}, j = 1, t) = 1 - \lambda_t / \lambda_{t+1} < 0$$

Estimation

Estimation Framework

- Suppose the data comprise N observations of the state variables and decisions denoted by $\left\{d_{nt_n}, x_{nt_n}, x_{n,t_n+1}\right\}_{n=1}^{N}$ sampled within a time frame of $t \in \{1, \ldots, S\}$.
- For expositional simplicity suppose the probability of sampling each $x \in \{1, \ldots, X\}$ in $t \in \{1, \ldots, S\}$ is strictly positive.
- M separate instances of finite dependence within that time frame.
- Say each pair of choices includes the first choice.
- Label the M paths by (j_m, x_m, t_m, ρ_m) for $m \in \{1, \ldots, M\}$.
- Assume:
  - $g(\epsilon_t)$ is known.
  - $\theta \equiv (\theta_1, \ldots, \theta_K) \in \Theta$, a closed convex set in $\mathbb{R}^K$.
  - $u_{jt}(x) \equiv \mu_{jt}(x, \theta)$, where $\mu_{jt}(x, \theta)$ is known function.
  - M instances of finite dependence suffice for identification.

Estimation

The reduced form for a minimum distance (MD) estimator

- For all $(t, x, j) \in \{1, \ldots, S\} \times \{1, \ldots, X\}\, t \in \{1, \ldots, J\}$:
  - define

$$\widehat{p}_{jt}(x) \equiv \frac{\sum_{n=1}^{N} 1\{d_{nt_nj} = 1\} 1\{t_n = t\} 1\{x_{nt_n} = x\}}{\sum_{n=1}^{N} 1\{t_n = t\} 1\{x_{nt_n} = x\}}$$

  - estimate the $XJ\mathcal{T}$ CCP vector $p \equiv (p_{11}(1), \ldots, p_{JS}(X))'$ with $\widehat{p}$ formed from $\widehat{p}_{jt}(x)$.
  - Also estimate $f_{jt}(x)$ with $\widehat{f}_{jt}(x)$ in this first stage, for example with a cell estimator (similar to the CCP estimator).

Estimation

The MD estimator

- Define $y\left(p,f\right)\equiv\left(y_1\left(p,f\right),\ldots,y_M\left(p,f\right)\right)'$ where:

$$y_m\left(p,f\right)\equiv\psi_1\left[p_{t_m}\left(x_m\right)\right]-\psi_{j_m}\left[p_{t_m}\left(x_m\right)\right]$$
$$+\sum_{\tau=t_m+1}^{t_m+\rho_m}\sum_{k=1}^{J}\sum_{x_\tau=1}^{X}\beta^{\tau-t_m}\psi_k\left[p_\tau\left(x_\tau\right)\right]\left[\begin{array}{l}\omega_{k\tau}\left(x_\tau,1,t_m\right)\kappa_\tau\left(x_\tau|1,x_m,t_m\right)-\\ \omega_{k\tau}\left(x_\tau,j_m,t_m\right)\kappa_\tau\left(x_\tau|j_m,x_m,t_m\right)\end{array}\right]$$

and $Z(p,f,\theta)\equiv\left(Z_1(p,f,\theta),\ldots,Z_M(p,f,\theta)\right)'$ where:

$$Z_m(p,f,\theta)\equiv\mu_{j_m,t_m}(x_m,\theta)$$
$$-\sum_{\tau=t_m+1}^{t_m+\rho_m}\sum_{k=1}^{J}\sum_{x_\tau=1}^{X}\beta_{k\tau}^{\tau-t_m}\mu_{k\tau}(x_\tau,\theta)\left[\begin{array}{l}\omega_{k\tau}\left(x_\tau,1,t_m\right)\kappa_\tau\left(x_\tau|1,x_m,t_m\right)-\\ \omega_{k\tau}\left(x_\tau,j_m,t_m\right)\kappa_\tau\left(x_\tau|j_m,x_m,t_m\right)\end{array}\right]$$

- For any M dimensional positive definite matrix W define:

$$\widehat{\theta}\equiv\arg\min_{\theta}\left[y\left(\widehat{p},\widehat{f}\right)-Z\left(\widehat{p},\widehat{f},\theta\right)\right]'W\left[y\left(\widehat{p}\right)-Z\left(\widehat{p},\widehat{f},\theta\right)\right]\qquad(9)$$

Estimation

Properties of the MD estimator

- If (θ_0, f) induces p, then $y(p, f) = Z(p, f, \theta_0)$.
- Hence $\widehat{\theta}$ is $\sqrt{N}$ consistent and asymptotically normal.
- Suppose $W = \widehat{W}$, a consistent estimate of the inverse of the asymptotic covariance matrix of $\left(\widehat{p}', \widehat{f}'\right)'$.
- In this case the asymptotic covariance matrix of $\widehat{\theta}$ is:

$$\left[Z\left(\widehat{p}, \widehat{f}, \theta\right)/\partial\theta' \, \widehat{W} Z\left(\widehat{p}, \widehat{f}, \theta\right)/\partial\theta\right]^{-1}$$

- If W is diagonal, (9) reduces to nonlinear least squares (NLS).
- Suppose $\mu_{jt}(x, \theta)$ is linear in θ and β is known. If $W = \widehat{W}$ then:

$$\widehat{\theta} = \left\{\frac{\partial Z\left(\widehat{p}, \widehat{f}, \theta\right)}{\partial\theta}' \widehat{W}\left[\partial Z\left(\widehat{p}, \widehat{f}, \theta\right)/\partial\theta\right]\right\}^{-1} \frac{\partial Z\left(\widehat{p}, \widehat{f}, \theta\right)}{\partial\theta}' \widehat{W} y\left(\widehat{p}, \widehat{f}\right)$$

One-Period Dependence in Optimization Problems

Approach

- *Guess and verify* is typically used to establish finite dependence.
- There is however a systematic way for determining finite period dependence.
- The algorithm iterates between two procedures that:
  1. checks counterfactual outcomes arising from deterministic choices that might either induce or rule out finite dependence.
  2. lists the elements of a matrix to determine its rank.
- Much of the intuition for this algorithm can be conveyed by analyzing one period dependence where there are only two choices.
- Note it is easier to establish one-period dependence because there are no intermediate decisions between the initial choice and the choice of weights that generate finite dependence.

One-Period Dependence in Optimization Problems

Equations determining one period dependence in a model with two choices

- When there are only 2 choices, one-period dependence specializes to:

$$\begin{aligned} & \kappa_{t+2}(x_{t+2}|2,x_t,t) \\ \equiv & \sum_{x_{t+1}=1}^{X}\sum_{k=1}^{J=2} f_{k,t+1}(x_{t+2}|x_{t+1})\omega_{k.t+1}(x_{t+1},2,t)\, f_{2t}(x_{t+1}|x_t) \\ = & \sum_{x_{t+1}=1}^{X}\sum_{k=1}^{J=2} f_{k,t+1}(x_{t+2}|x_{t+1})\omega_{k.t+1}(x_{t+1},1,t)\, f_{1t}(x_{t+1}|x_t) \\ \equiv & \kappa_{t+2}(x_{t+2}|1,x_t,t) \end{aligned} \tag{10}$$

- The weights sum to one, implying:

$$\omega_{1.t+1}(1,x_{t+1},t) = 1 - \omega_{2.t+1}(1,x_{t+1},t)$$

and $\kappa_{t+2}(x_{t+2}|2,x_t,t)$ is a weight, so:

$$\kappa_{t+2}(X|j,x_t,t) = 1 - \sum_{x_{t+2}=1}^{X-1} \kappa_{t+2}(x_{t+2}|2,x_t,t)$$

One-Period Dependence in Optimization Problems

Counting the potential equations and unknowns

- Substituting out $\omega_{1.t+1}\left(x_{t+1}, 1, t\right)$ we rewrite (10) as:

$$\sum_{x_{t+1}=1}^{X} \left\{ \begin{array}{c} \left[f_{2,t+1}(x_{t+2}|x_{t+1}) - f_{1,t+1}(x_{t+2}|x_{t+1})\right] \\ \times \left[\begin{array}{c} \omega_{2.t+1}\left(x_{t+1}, 2, t\right) f_{2t}(x_{t+1}|x_t) \\ -\omega_{2.t+1}\left(x_{t+1}, 1, t\right) f_{1t}(x_{t+1}|x_t) \end{array} \right] \end{array} \right\}$$

$$= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) \left[f_{1t}(x_{t+1}|x_t) - f_{2t}(x_{t+1}|x_t)\right] \quad (11)$$

  for all $x_{t+2} \in \{1, 2, \ldots, X-1\}$.

- Nominally this is a linear system:
  - comprising $X-1$ equations
  - in $2X$ weights, $\omega_{2.t+1}\left(1, k, t\right), \ldots, \omega_{2.t+1}\left(X, k, t\right)$ for $k \in \{1, 2\}$.
  - with $X-1$ more unknowns than equations.
- Therefore one-period finite dependence holds if and only if a rank condition for the system is satisfied.

One-Period Dependence in Optimization Problems

Reducing the dimensions of the problem

- However this only upper bounds the number of weights and equations that need to be considered. Define:

$$\begin{aligned}\mathcal{A}_{j,t+1}(x_t) &\equiv \{x \in \mathbb{X} : f_{jt}(x|x_t) > 0\} \\ \mathcal{A}_{t+2}(x_t) &\equiv \left\{ \begin{array}{l} x \in \mathbb{X} : f_{k,t+1}(x|x') > 0 \\ \text{for } x' \in \mathcal{A}_{1,t+1} \bigcup \mathcal{A}_{2,t+1} \text{ and } k \in \{1,2\} \end{array} \right\}\end{aligned}$$

- If $x \notin \mathcal{A}_{j,t+1}(x_t)$ then $\omega_{k.t+1}(t,x,j)$ is irrelevant

$\Longrightarrow$ reducing the number of weights to solve the system.

- If $x \notin \mathcal{A}_{t+2}(x_t)$ then $\kappa_{t+2}(x|2,x_t,t) = \kappa_{t+2}(x|1,x_t,t) = 0$

$\Longrightarrow$ reducing the number of equations to be satisfied.

- Thus the system (11) reduces to $A_{t+2} - 1$ equations linear in $A_{1,t+1} + A_{2,t+1}$ weights, where:

$$A_{j,t+1} \equiv \sum_{x=1}^{X} \mathbf{1}\{x \in \mathcal{A}_{j,t+1}(x_t)\} \qquad A_{t+2} \equiv \sum_{x=1}^{X} \mathbf{1}\{x \in \mathcal{A}_{t+2}(x_t)\}$$

One-Period Dependence in Optimization Problems

Deriving a matrix representation

- We can incorporate these two features into the system of equations given by (11). Denote:
  - the $A_{j,t+1}$ dimensional vector of nonzero probabilities in the string $f_{jt}(1|x_t), \ldots, f_{jt}(X|x_t)$ by

$$\mathsf{K}_{j,t+1}(\mathcal{A}_{j,t+1})$$

  - the $\mathcal{A}_{j,t+1}$ by $\mathcal{A}_{t+2} - 1$ matrix, the first $A_{t+2} - 1$ columns of the $A_{j,t+1} \times A_{t+2}$ transition matrix from $\mathcal{A}_{j,t+1}$ to $\mathcal{A}_{t+2}$ when choice k is made in period $t+1$ (containing elements $f_{k,t+1}(x'|x)$, where $x \in \mathcal{A}_{j,t+1}$ and $x' \in \mathcal{A}_{t+2}$) by

$$\mathsf{F}_{k,t+1}(\mathcal{A}_{j,t+1})$$

  - the $A_{j,t+1}$ dimensional vector of weights on each of the attainable states, given initial choice j at $t+1$ for setting $d_{2t} = 1$, comprising elements $\omega_{t+1}(x, j)$ for each $x \in \mathcal{A}_{j,t+1}$ by

$$\Omega_{t+1}\left(\mathcal{A}_{j,t+1}, j\right)$$

One-Period Dependence in Optimization Problems

Rank condition

- Also let:

$$\mathcal{K}_{t+1} \equiv \begin{bmatrix} \mathsf{F}_{1,t+1}(\mathcal{A}_{1,t+1}) \\ -\mathsf{F}_{1,t+1}(\mathcal{A}_{2,t+1}) \end{bmatrix}' \begin{bmatrix} \mathsf{K}_{1,t+1}(\mathcal{A}_{1,t+1}) \\ \mathsf{K}_{2,t+1}(\mathcal{A}_{2,t+1}) \end{bmatrix}$$

$$\mathsf{H}_{t+1} \equiv \begin{bmatrix} \mathsf{F}_{2,t+1}(\mathcal{A}_{2,t+1}) - \mathsf{F}_{1,t+1}(\mathcal{A}_{2,t+1}) \\ \mathsf{F}_{1,t+1}(\mathcal{A}_{1,t+1}) - \mathsf{F}_{2,t+1}(\mathcal{A}_{1,t+1}) \end{bmatrix}$$

- Substituting these transformations into (11), one period dependence holds if and only if there exists an $(A_{1,t+1} + A_{2,t+1})$ vector of unknowns denoted by D_{t+1} solving:

$$\mathcal{K}_{t+1} = \mathsf{H}_{t+1} \begin{bmatrix} \Omega_{t+1}\left(\mathcal{A}_{j,t+1}, 2\right) \circ \mathsf{K}_{2,t+1}(\mathcal{A}_{2,t+1}) \\ \Omega_{t+1}\left(\mathcal{A}_{j,t+1}, 1\right) \circ \mathsf{K}_{1,t+1}(\mathcal{A}_{1,t+1}) \end{bmatrix} \equiv \mathsf{H}_{t+1}\mathsf{D}_{t+1} \tag{12}$$

  where $\circ$ means element-by-element multiplication.
- A solution to (12) for D_{t+1} exists if and only if the rank of H_{t+1} equals the rank of $\mathsf{H}^*_{t+1} \equiv \left[\mathcal{K}_{t+1} \vdots \mathsf{H}_{t+1}\right]$.