2 Discrete Choices

Robert A. Miller

Leverhulme Visiting Professor
University College London
&
Richard M. Cyert and Morris DeGroot Professor of Economics and Statistics
Carnegie Mellon University

May 2023

- **When should choices be modeled as continuous versus discrete?**

1. *How comparable are the different choices?*
   - a bigger slice of cake with more icing (continuous)
   - versus apple or orange (discrete)
2. *Are they lumpy or not?*
   - participate or not in the labor force (discrete)
   - versus how many hours (continuous)
3. *What about assumptions on the unobserved variables?*
   - often distributional independence (discrete)
   - versus conditional mean independence (continuous)

- In addition discrete choices might be:
  - *ordered* (fuel stops on a road trip)
  - *categorical* (choice of vacation destination)

A Class of Dynamic Discrete Choice Markov Models

Discrete time and finite choice sets

- Let $t \in \{1, \ldots, T\}$ denote the time period for some $T \leq \infty$.
- Each period the individual chooses amongst J actions. Write:
  - $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ and $\sum_{j=1}^{J} d_{jt} = 1$ where:

$$d_{jt} = \begin{cases} 1 \text{ if action } j \text{ is taken at } t \\ 0 \text{ if not} \end{cases}$$

- The random variables influencing this decision include:
  - $x_t \in \{1, \ldots, X\}$ for some finite positive integer X for each t.
  - $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ where $\epsilon_{jt} \in \mathbb{R}$ for all (j, t).
- The data comprise a panel of individuals on (d_t, x_t).
- Let $g_{t,x,\epsilon}(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t)$:
  - denote mixed density function for $(x_{t+1}, \epsilon_{t+1})$ conditional on (x_t, ϵ_t)
  - satisfy the conditional independence assumption:

$$g_{t,j,x,\epsilon}(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t) = g_{t+1}(\epsilon_{t+1} | x_{t+1}) f_{jt}(x_{t+1} | x_t)$$

- Denoting the discount factor by $\beta \in (0,1)$, the current payoff from taking action j at t given (x_t, ϵ_t) by:

$$u_j(x_t) + \epsilon_{jt}$$

- To ensure a transversality condition is satisfied, assume $\left\{ \int \max \left\{ |\epsilon_{1t}|, \ldots, |\epsilon_{Jt}| \right\} g_t\left(\epsilon_t | x_t\right) d\epsilon_t \right\}_{t=1}^{T}$ is a bounded sequence.
- At the beginning of each period t:
  - the agent observes the realization (x_t, ϵ_t)
  - chooses d_t to sequentially maximize:

$$E\left\{ \sum_{\tau=t}^{T} \sum_{j=1}^{J} \beta^{\tau-1} d_{j\tau} \left[u_{j\tau}(x_\tau) + \epsilon_{j\tau} \right] | x_t, \epsilon_t \right\} \tag{1}$$

  where the expectation is taken over future realized values $x_{t+1}, \ldots, x_T$ and $\epsilon_{t+1}, \ldots, \epsilon_T$ conditional on (x_t, ϵ_t).

- Denote the optimal decision rule at t as $d_t^o(x_t, \epsilon_t)$, with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$ and define:

$$V_t(x_t) \equiv E\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t-1}d_{j\tau}^o\left(x_\tau, \epsilon_\tau\right)\left(u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right)\right\}$$

- The conditional value function, $v_{jt}(x_t)$, is defined as:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta\sum_{x=1}^{X}V_{t+1}(x)f_{jt}\left(x|x_t\right)$$

- Integrating $d_{jt}^o(x_t, \epsilon)$ over $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_J)$:

$$p_{jt}(x_t) \equiv E\left[d_{jt}^o\left(x_t, \epsilon\right)|x_t\right] = \int d_{jt}^o\left(x_t, \epsilon\right)g_t\left(\epsilon\,|x_t\right)d\epsilon$$

Inversion

Differences in conditional valuation functions

- The starting point for our analysis is to define differences in the conditional valuation functions as:

$$\Delta v_{jkt}\left(x\right) \equiv v_{jt}\left(x\right) - v_{kt}\left(x\right)$$

- Although there are $J\left(J-1\right)$ differences all but $\left(J-1\right)$ are linear combinations of the $\left(J-1\right)$ basis functions.
- For example setting the basis functions as:

$$\Delta v_{jt}\left(x\right) \equiv v_{jt}\left(x\right) - v_{Jt}\left(x\right)$$

then clearly:

$$\Delta v_{jkt}\left(x\right) = \Delta v_{jt}\left(x\right) - \Delta v_{kt}\left(x\right)$$

- Without loss of generality we focus on this particular basis function.

Inversion

Each CCP is a mapping of differences in the conditional valuation functions

- Using the definition of $\Delta v_{jt}(x)$:

$$
\begin{aligned}
p_{jt}(x) &\equiv \int d_{jt}^{o}(x,\epsilon)\, g_t(\epsilon\,|x)\, d\epsilon \\
&= \int I\{\epsilon_k \le \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall\ k \neq j\}\, g_t(\epsilon\,|x)\, d\epsilon \\
&= \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{1t}(x)} \cdots \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{J-1,t}(x)} \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)} g_t(\epsilon\,|x)\, d\epsilon
\end{aligned}
$$

- Noting $g_t(\epsilon\,|x) \equiv \partial^J G_t(\epsilon\,|x)/\partial\epsilon_1,\ldots,\partial\epsilon_J$, integrate over $(\epsilon_1,\ldots,,\epsilon_{j-1},\epsilon_{j+1}\ldots,\epsilon_J)$.
- Denoting $G_{jt}(\epsilon\,|x) \equiv \partial G_t(\epsilon\,|x)/\partial\epsilon_j$, yields:

$$
p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \ldots \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x \right) d\epsilon_j
$$

Inversion

There are as many CCPs as there are conditional valuation functions

- For any vector $J-1$ dimensional vector $\delta \equiv (\delta_1, \ldots, \delta_{J-1})$ define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}\left(\epsilon_j + \delta_j - \delta_1, \ldots, \epsilon_j, \ldots, \epsilon_j + \delta_j \,|x\right) d\epsilon_j$$

- We interpret $Q_{jt}(\delta, x)$ as the probability taking action j in a static random utility model (RUM) where the payoffs are $\delta_j + \epsilon_j$ and the probability distribution of disturbances is given by $G_t(\epsilon \,|x)$.
- It follows from the definition of $Q_{jt}(\delta, x)$ that:

$$0 \leq Q_{jt}(\delta, x) \leq 1 \text{ for all } (j, t, \delta, x) \text{ and } \sum_{j=1}^{J-1} Q_{jt}(\delta, x) \leq 1$$

- In particular the previous slide implies that for any given (j, t, x):

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x\right) d\epsilon_j \equiv Q_{jt}(\Delta v_t(x), x)$$

Theorem (Inversion)

For each (t, δ, x) *define:*

$$Q_t(\delta, x) \equiv \left(Q_{1t}(\delta, x), \ldots Q_{J-1,t}(\delta, x)\right)'$$

Then the vector function $Q_t(\delta, x)$ *is invertible in* δ *for each* (t, x).

- Note that $p_{Jt}(x) = Q_{Jt}(\Delta v_t, x)$ is a linear combination of the other equations in the system because $\sum_{k=1}^{J} p_k = 1$.
- Let $p \equiv (p_1, \ldots, p_{J-1})$ where $0 \leq p_j \leq 1$ for all $j \in \{1, \ldots, J-1\}$ and $\sum_{j=1}^{J-1} p_j \leq 1$. Denote the inverse of $Q_{jt}(\Delta v_t, x)$ by $Q_{jt}^{-1}(p, x)$.
- The inversion theorem implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}[p_t(x), x] \\ \vdots \\ Q_{J-1,t}^{-1}[p_t(x), x] \end{bmatrix}$$

- We can use the Inversion Theorem to:
    1. provide empirically tractable representations of the conditional value functions.
    2. analyze identification in dynamic discrete choice models.
    3. provide convenient parametric forms for the density of ϵ_t that generalize the Type 1 Extreme Value distribution.
    4. introduce new methods for incorporating unobserved state variables.

Corollaries of the Inversion Theorem

Identifying the policy function

- From the definition of the optimal decision rule, and then appealing to the inversion theorem:

$$
\begin{aligned}
d^o_{jt}\left(x_t, \epsilon_t\right) &= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x) - v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq \begin{array}{l} v_{jt}(x) - v_{Jt}(x_t) \\ -\left[v_{kt}(x) - v_{Jt}(x_t)\right] \end{array}\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq \Delta v_{jt}(x) - \Delta v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq Q^{-1}_{jt}\left[p_t(x), x\right] - Q^{-1}_{kt}\left[p_t(x), x\right]\right\}
\end{aligned}
$$

- If $G_t\left(\epsilon \,|x\right)$ is known and the data generating process (DGP) is (x_t, d_t), then $p_t(x)$ and hence $d^o_t\left(x_t, \epsilon_t\right)$ are identified.

- Define the conditional value function correction as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- In stationary settings, we drop the t subscript and write:

$$\psi_j(x) \equiv V(x) - v_j(x)$$

- Suppose that instead of taking the optimal action she committed to taking action j instead. Then the expected lifetime utility would be:

$$v_{jt}(x_t) + E_t\left[\epsilon_{jt} \,|x_t\right]$$

  so committing to j before ϵ_t is revealed entails a loss of:

$$V_t(x_t) - v_{jt}(x_t) - E_t\left[\epsilon_{jt} \,|x_t\right] = \psi_{jt}(x) - E_t\left[\epsilon_{jt} \,|x_t\right]$$

- For example if $E_t\left[\epsilon_t \,|x_t\right] = 0$, the loss simplifies to $\psi_{jt}(x)$.

- From their respective definitions:

$$
\begin{aligned}
& V_t(x) - v_{it}(x) \\
= \; & \sum_{j=1}^{J} \left\{ p_{jt}(x) \left[v_{jt}(x) - v_{it}(x) \right] + \int \epsilon_{jt} d_{jt}^{o} \left(x_t, \epsilon_t \right) g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t \right\}
\end{aligned}
$$

- But:

$$
v_{jt}(x) - v_{it}(x) = Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{it}^{-1} \left[p_t(x), x \right]
$$

and

$$
\begin{aligned}
& \int \epsilon_{jt} d_{jt}^{o} \left(x, \epsilon_t \right) g \left(\epsilon_t \left| x \right. \right) d\epsilon_t \\
= \; & \int \prod_{k=1}^{J} 1 \left\{ \begin{array}{l} \epsilon_{kt} - \epsilon_{jt} \\ \leq Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{kt}^{-1} \left[p_t(x), x \right] \end{array} \right\} \epsilon_{jt} g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t
\end{aligned}
$$

- Therefore $\psi_{it}(x) \equiv V_t(x) - v_{it}(x)$ is identified if $G_t \left(\epsilon \left| x \right. \right)$ is known and (x_t, d_t) is the DGP.

Conditional Valuation Function Representation

Telescoping one period forward

- From its definition:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}\left(x_{t+1} | x_t\right)$$

- Substituting for $V_{t+1}(x_{t+1})$ using conditional value function correction we obtain for any k:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} \left[v_{k,t+1}(x) + \psi_{k,t+1}(x)\right] f_{jt}\left(x | x_t\right)$$

- We could repeat this procedure ad infinitum, substituting in for $v_{k,t+1}(x)$ by using the definition for $\psi_{kt}(x)$.

Conditional Valuation Function Representation

Recursively defining the distribution of future state variables

- To formalize this idea, consider a random sequence of weights from t to T which begins with $\omega_{jt}(x_t, j) = 1$.
- For periods $\tau \in \{t+1, \ldots, T\}$, the choice sequence maps x_τ and the initial choice j into

$$\omega_\tau(x_\tau, j) \equiv \{\omega_{1\tau}(x_\tau, j), \ldots, \omega_{J\tau}(x_\tau, j)\}$$

where $\omega_{k\tau}(x_\tau, j)$ may be negative or exceed one but:

$$\sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) = 1$$

.

- The weight of state $x_{\tau+1}$ conditional on following the choices in the sequence is recursively defined by $\kappa_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and for $\tau = t+1, \ldots, T$:

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) f_{k\tau}(x_{\tau+1}|x_\tau)\kappa_{\tau-1}(x_\tau|x_t, j)$$

Conditional Valuation Function Representation

Theorem 1 of Arcidiacono and Miller (2011)

Theorem (Representation)

For all periods, states and choices (t, x_t, j),*and any weights* $\omega_\tau(x_\tau, j)$,
$v_{jt}(x_t) =$

$$u_{jt}(x_t) + \sum_{\tau=t+1}^{T} \sum_{k=1}^{J} \sum_{x=1}^{X} \beta^{\tau-t} \left[u_{k\tau}(x) + \psi_k[p_\tau(x)] \right] \omega_{k\tau}(x, j) \kappa_{\tau-1}(x|x_t, j)$$

- The theorem yields an alternative expression for $v_{jt}(x_t)$ that dispenses with recursive maximization.
- Intuitively, the individuals have already solved their optimization problem, so their decisions, as reflected in their CCPs, are informative of their value functions.

Generalized Extreme Values

Definition

- Are there tractable distributions $G_t(\epsilon | x)$ aside from the Type 1 Extreme Value?
- Suppose $G(\epsilon)$ factors into two independent distributions, one a nested logit, and the other any GEV distribution.
- Let $\mathcal{J}$ denote the set of choices in the nest and denote the other distribution by $G_0(Y_1, Y_2, \ldots, Y_K)$ let K denote the number of choices that are outside the nest.
- Then:

$$G(\epsilon) \equiv G_0(\epsilon_1, \ldots, \epsilon_K) \exp\left[-\left(\sum_{j \in \mathcal{J}} \exp\left[-\epsilon_j / \sigma\right]\right)^{\sigma}\right]$$

- The correlation of the errors within the nest is given by $\sigma \in [0, 1]$ and errors within the nest are uncorrelated with errors outside the nest. When $\sigma = 1$, the errors are uncorrelated within the nest, and when $\sigma = 0$ they are perfectly correlated.

Lemma

For the nested logit $G(\epsilon_t)$ defined above:

$$\psi_j(p) = \gamma - \sigma \ln(p_j) - (1-\sigma)\ln\left(\sum_{k\in\mathcal{J}} p_k\right)$$

- Note that $\psi_j(p)$ only depends on the conditional choice probabilities for choices that are in the nest: the expression is the same no matter how many choices are outside the nest or how those choices are correlated.
- Hence, $\psi_j(p)$ will only depend on $p_{j'}$ if ϵ_{jt} and $\epsilon_{j't}$ are correlated. When $\sigma = 1$, ϵ_{jt} is independent of all other errors and $\psi_j(p)$ only depends on p_j.

- The optimization model is fully characterized by the time horizon, the utility flows, the discount factor, the transition matrix of the observed state variables, and the distribution of the unobserved variables, summarized with the notation (T, β, f, g, u).
- The data comprise observations for a real or synthetic panel on the observed part of the state variable, x_t, and decision outcomes, d_t.
- Following most of the empirical work in this area we consider identification when (T, β, f, g) are assumed to be known.
- Thus the goal is to identify u from (x_t, d_t) when (T, β, f, g) is known.

Theorem (Identification)

For all j, t, and x:

$$u_{jt}(x) = u_{1t}(x) + \psi_{1t}(x) - \psi_{jt}(x) + \sum_{\tau=t+1}^{T} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} [u_{1\tau}(x_\tau) + \psi_{1t}(x_\tau)] \times \\ [\kappa_{\tau-1}(x_\tau|x,1) - \kappa_{\tau-1}(x_\tau|x,j)] \end{array} \right\} \tag{2}$$

In stationary models, define $\Psi_j \equiv \left[\psi_j(1) \ldots \psi_j(X)\right]'$, *and for all j:*

$$u_j = \Psi_1 - \Psi_j - u_1 + \beta(F_1 - F_j)[I - \beta F_1]^{-1}(\Psi_1 + u_1) \tag{3}$$

- If (T, β, f, g) is known, and if a payoff, say the first, is also known for every state and time, then u is (exactly) identified.

American Dream Delayed (Khounzhina and Miller, 2022)

Applying the framework

- The average age of a first-time home buyer was about 28 years old in the 1970s, about 30 in the 1990's, and is now about 32.5.
- This increase coincided with postponing marriage and fertility; the average age of mother at first birth rose from 22 forty years ago to 24 two decades ago, and is currently about 26.
- In contrast female labor-force participation rose from 48 percent in 1975, to 74 percent in 1995 and 76 percent in 2015, hours worked following a similar pattern.
- What role did the following four economic factors play?
  1. Real wages to females rose.
  2. Females became more educated.
  3. The real interest rate declined.
  4. Housing prices rose and then fell.

- Denote by:
  - $b_t \in \{0, 1\}$, where $b_t = 1$ if a child is born at time t.
  - $c_t \in \mathcal{R}$ denotes nonhousing consumption, a continuous choice.
  - $l_t \in \{0, 1\}$, where $l_t = 1$ means female works at time t.
  - $h_t \in \{0, 1\}$, where $h_t = 1$ means first home is purchased at t.
- If $h_t = 1$ then $h_\tau = 0$ for $\tau \in \{t + 1, \dots, T\}$.
- Define homeownership by $h_t^* \equiv \sum_{\tau=1}^{t-1} h_\tau$. Then there are:
  - eight (b_t, l_t, h_t) discrete choices combinations if $h_t^* = 0$.
  - effectively four (b_t, l_t) combinations if $h_t^* = 1$.
- We label each possible choice permutation by $d_{jt} \in \{0, 1\}$ where:
  - $j \in \{0, \dots, 7\}$ and if $h_t^* = 1$ then $j \in \{0, \dots, 3\}$.
  - $\sum_{j=0}^{7} d_{j\tau} = 1$ and $\sum_{j=0}^{3} d_{j\tau} = 1$ if $h_t^* = 1$.

- We model household lifetime utility from t onwards as:

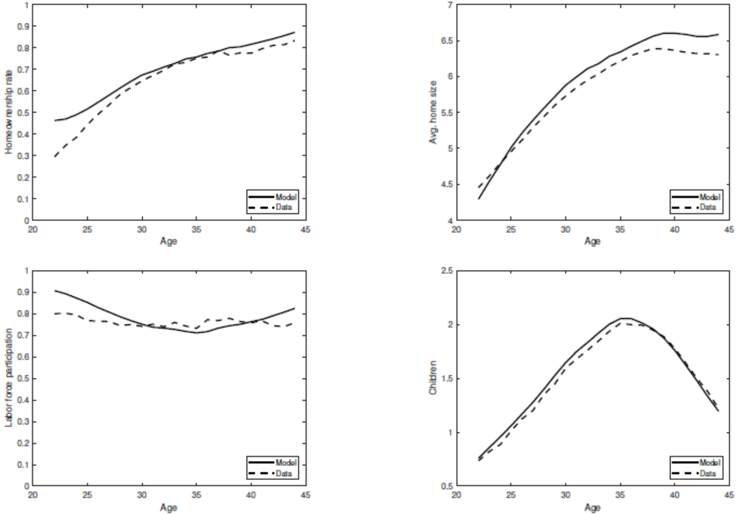
$$-\sum_{\tau=t}^{\infty}\sum_{j=0}^{7}\beta^{\tau-t}d_{j\tau}\exp\left(h_{\tau}u_{\tau}^{h}+b_{\tau}u_{j\tau}^{b}+l_{\tau}u_{\tau}^{l}-\rho c_{\tau}-\epsilon_{j\tau}\right)$$

  where j indexes the discrete choices at τ and:

  - β denotes the subjective discount factor.
  - u_{τ}^{h} indexes expected lifetime utility from purchasing first home.
  - u_{τ}^{b} indexes net expected lifetime utility of raising a child.
  - u_{τ}^{l} indexes the current utility of current leisure.
  - ρ is the constant absolute risk aversion parameter.
  - $\epsilon_{j\tau}$ is a period τ choice-specific disturbance with *iid* density $g\left(\epsilon_{j\tau}\right)$.

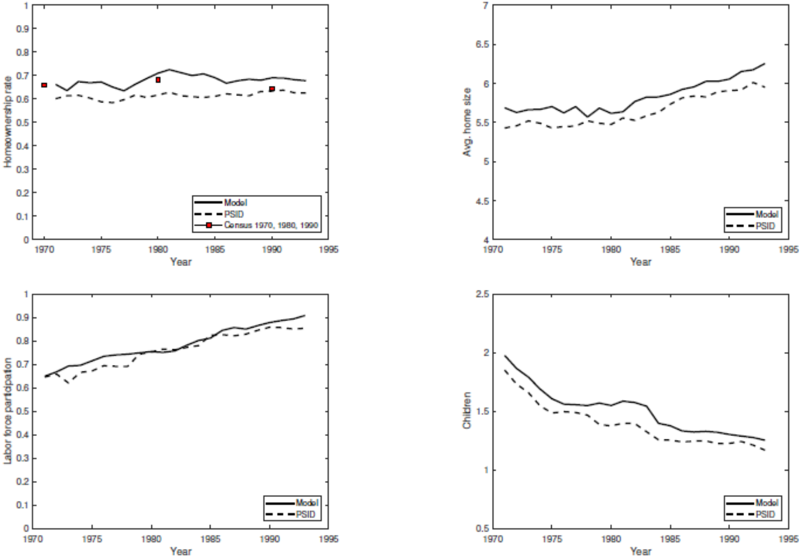
American Dream Delayed

Cross-section fit (Figure 5 Khorunzhina and Miller 2022)



American Dream Delayed

Time-series fit (Figure 5 Khorunzhina and Miller 2022)



	Average age at		LFP (%)	
	first child	first homeownership	before age 35	after age 35
Benchmark in 1971	22.9	28.0	78	67
Benchmark in 1991	23.5	27.8	85	76
A. Wage as in 1991	21.6	29.3	87	85
B. Education level as in 1991	23.7	27.5	79	67
C. House prices as in 1991	22.9	28.4	78	68
D. Interest rate as in 1991	23.3	27.0	72	55