1 Continuous Choices

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Introduction

Using models to summarize the data

- Two fundamental challenges facing social scientists are:
  1. Large data sets *(with many observations and variables)* are **too complex to thoroughly digest in raw form**.
  2. Inferences about **a population from a sample** requires assumptions.
- These challenges rationalize building a model:
  1. a **collection of assumptions** relating the data to the population
  2. which produces estimates from the data that **characterize** the population
- Structural econometric models are distinctive because **the data generating process is derived from economic theory**.
- They are useful in predicting:
  1. **counterfactuals** *(adjusting to hypothetical policy or technological innovations)*
  2. the behavior of **under represented segments** in the population.
- How might we construct estimators derived from continuous choice models of **competitive equilibrium** and **auctions**?

Law of One Price (Debreu, 1959)

Competitive equilibrium

- **Competitive equilibrium** is the bedrock of economics:
  - Consumers reveal their preferences through their choices (*three axioms supporting consumer choice theory*).
  - Given the price of each commodity, consumers and producers buy or sell as many units as they wish (*individual optimization*).
  - At those prices the market for each commodity clears, supply matching demand (*existence of equilibrium*).
- Most econometric analysis also make several auxiliary assumptions.

1. The expected utility hypothesis holds.
2. Subjective beliefs match probability transitions.
3. Preferences are time additively separable up to a finite vector of human capital, habit persistence and nondurables.
4. Current utility payoffs are sufficiently smooth that individual agents only require a small number of securities to achieve the equilibrium allocations of the complete markets.

- Suppose there are J financial securities.
- Let p_{tj} denote the price of the j^{th} security in period t consumption units, and $q_{t-1,j}$ the amount a consumer owns at the beginning of the period.
- Let r_{tj} denote the real return on assets purchased in period $t-1$.
- The investor's budget constraint is:

$$c_t + \sum_{j=1}^{J} p_{tj} q_{tj} \leq \sum_{j=1}^{J} r_{tj} p_{t-1,j} q_{t-1,j}$$

- At t the consumer maximizes a concave objective function with linear constraints, choosing $(q_{s1}, ..., q_{sJ})$ to maximize:

$$u(c_t) + E_t \left[\sum_{s=t+1}^{T} \beta^{s-t} u(c_s) \right]$$

subject to the sequence of all the future budget constraints.

- Nonsatiation guarantees:

$$c_t = \sum_{j=1}^{J} \left(r_{tj} p_{t-1,j} q_{t-1,j} - p_{tj} q_{tj} \right)$$

- The interior first order condition (FOC) for each $k \in \{1, \ldots, J\}$ requires:

$$\begin{aligned} & p_{tk} u' \left(\sum_{j=1}^{J} \left(r_{tj} p_{t-1,j} q_{t-1,j} - p_{tj} q_{tj} \right) \right) \\ \geq \quad & E_t \left[p_{tk} r_{t+1,k} \beta u' \left(\sum_{j=1}^{J} \left(r_{t+1,j} p_{tj} q_{tj} - p_{t+1,j} q_{t+1,j} \right) \right) \right] \end{aligned}$$

with equality holding if $q_{tj} > 0$.

- For any $r \times 1$ vector x_t belonging to the information set at t and all k:

$$0 = E_t \left[r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \right] = E \left[r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \,| x_t \right]$$

  and hence:

$$0 = E \left\{ x_t \left[r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \right] \right\}$$

- Given a sample of length T we can estimate the $1 \times l$ vector (β, α) for a parametrically defined utility function $u(c_t; \alpha)$ by solving:

$$0 = A_T \sum_{t=1}^{T} x_t \left[r_{t+1,k} \beta \frac{u'(c_{t+1}; \alpha)}{u'(c_t; \alpha)} - 1 \right]$$

  where A_T is an $l \times r$ weighting matrix.
- The large sample properties of this estimator are based on:
  1. large T.
  2. a stationary economy.

Complete Markets (Altug and Miller,1990)

A lifetime budget constraint

- An alternative approach appeals to a law of large numbers in the cross section
- Assume markets are *complete* and individuals are *atomless*:
  - There is a competitive market for every state of the world commodity defined on k and t.
  - Individual idiosyncratic risk is actuarially priced.
- These assumptions imply:
  - the budget set for household n is a single lifetime budget constraint,
  - rather than a sequence of period-specific budget constraints:

$$\sum_{t=0}^{T} \sum_{k=1}^{K} \prod_{s=1}^{t} (1 + r_s)^{-1} p_{tk} E_0 [c_{ntk}] \leq B_n \tag{1}$$

  where:
  - $c_{nt} \equiv (c_{nt1}, \ldots, c_{ntK})$ denote the consumption vector of n at t.
  - p_{tk} is the spot price of commodity k consumed in period t.
  - r_s is the one period interest rate in period s.

Complete Markets

Time additive preferences, maximization and the first order condition

- Suppose households obey the expected utility hypothesis, preferences taking the time additive form:

$$E_0 \left[\sum_{t=0}^{T} \beta^t u\left(c_{nt}, \epsilon_{nt}\right) \right] \tag{2}$$

  where $\epsilon_{nt} \equiv \left(\epsilon_{nt1}, \ldots, \epsilon_{ntK}\right)$ is an iid disturbance to its preferences.
- Household n chooses c_{nt} taking into account all future interest rates, spot prices, its budget constraint and its current disturbance ϵ_{nt}.
- Maximizing (2) subject to (1), the interior FOC for k is:

$$\beta^{t_n} u_k\left(c_{nt}, \epsilon_{nt}\right) \equiv \beta^{t_n} \frac{\partial u\left(c_{nt}, \epsilon_{nt}\right)}{\partial c_{ntk}} = \eta_n \prod_{s=1}^{t} \left(1 + r_s\right)^{-1} p_{tk} \tag{3}$$

  where η_n denote the Lagrange multiplier associated with (1).

Complete Markets

Marginal rates of substitution in equilibrium

- Temporarily dropping the subscript n, and setting $p_{t1} \equiv 1$, there are:
  1. $(K-1)(T+1)$ equations corresponding to the spot markets:

$$MRS_k(c_t) \equiv \frac{u_k(c_t)}{u_1(c_t)} = p_{tk}$$

  2. T equations pertaining to the numeraire that intertemporally balance consumption:

$$MRS(c_t, c_{t+1}) \equiv \frac{\beta u_1(c_{t+1})}{u_1(c_t)} = (1+r_t)^{-1}$$

- Given η_n, these $K(T+1)-1$ MRS conditions fully characterize an interior equilibrium consumption.

Complete Markets
Example

- For example suppose:

$$u\left(c_{nt}\right) \equiv \sum_{k=1}^{K} \frac{\exp\left(x_{nt}B_k + \epsilon_{ntk}\right)}{\alpha_k + 1} c_{ntk}^{\alpha_k+1}$$

- Focusing on the first two goods we have:

$$\begin{aligned} p_{t2} &= MRS_{t2}\left(c_{nt}\right) \\ &= \exp\left[x_{nt}\left(B_2 - B_1\right) + \epsilon_{nt2} - \epsilon_{nt1}\right] \frac{c_{nt2}^{\alpha_2}}{c_{nt1}^{\alpha_1}} \end{aligned}$$

Taking logarithms:

$$\begin{aligned} & \epsilon_{nt2} - \epsilon_{nt1} \\ = \;& x_{nt}\left(B_1 - B_2\right) + \alpha_1 \ln\left(c_{nt1}\right) - \alpha_2 \ln\left(c_{nt2}\right) + \ln p_{t2} \end{aligned}$$

- For any instrument vector z_{nt} satisfying:

$$E\left[\epsilon_{nt} \left| z_{nt}\right.\right] = 0$$

we have:

$$E\left\{z_{nt}\left[x_{nt}\left(B_1 - B_2\right) + \alpha_1 \ln\left(c_{nt1}\right) - \alpha_2 \ln\left(c_{nt2}\right) + \ln p_{t2}\right]\right\} = 0$$

- A GMM estimator now comes from setting

$$0 = A\sum_{n=1}^{N} z_{nt}\left[x_{nt}\left(B_1 - B_2\right) + \alpha_1 \ln\left(c_{nt1}\right) - \alpha_2 \ln\left(c_{nt2}\right) + \ln p_{t2}\right]$$

- The usual large sample properties apply.

- Similarly:

$$\left(1+r_t\right)^{-1}=\beta\exp\left[\left(x_{n,t+1}-x_{nt}\right)B_1+\epsilon_{n,t+1,1}-\epsilon_{nt1}\right]\left(\frac{c_{nt+1,1}}{c_{n,t+1,1}}\right)^{\alpha_1}$$

or in logarithmic form:

$$-\Delta\ln\left(1+r_t\right)-\ln\beta=\Delta x_{nt}B_1+\Delta\epsilon_{nt1}+\alpha_1\Delta\ln c_{nt1}$$

where $\Delta x_{nt}\equiv\ \left(x_{n,t+1}-x_{nt}\right)$ and so forth.

- If $E\left[\epsilon_{nt}\left|z_{nt}\right.\right]=0$ then:

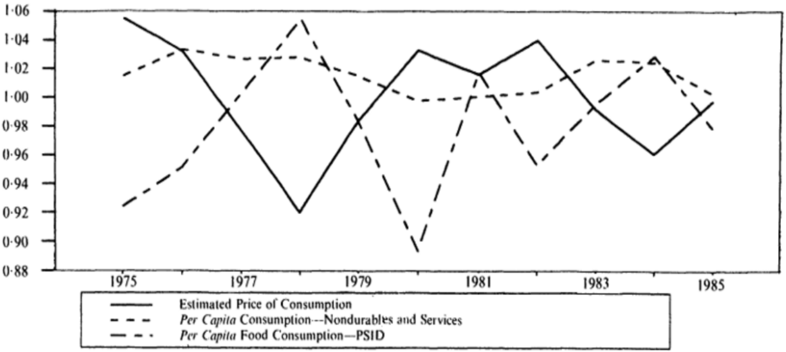
$$E\left\{z_{nt}\left[-\Delta\ln\left(1+r_t\right)-\ln\beta-\alpha_1\Delta\ln c_{nt1}-\Delta x_{nt}B_1\right]\right\}=0$$

- A GMM estimator with the usual large sample properties can be formed from the sample analogue.

Tracking Aggregates (Altug and Miller, 1998)

Allowing for heterogeneity when tracking aggregate shocks

- These results for a (slighly more sophisticated) model illustrates why this assumption of complete markets gives reasonable approxiations to aggregate trends:



An International Comparison (Miller and Sieg, 1997)

A model of male labor supply and housing demand

- The following notation applies to household n at time t:
  - l_{0nt} female leisure
  - l_{1nt} male leisure
  - h_{nt} housing services
  - x_{nt} observed demographics
- Current utility takes the form:

$$u\left(l_{0nt}, l_{1nt}, h_{nt}, , x_{nt}\right) \equiv \alpha_0^{-1} \exp\left(x_{nt} B_0 + \epsilon_{0nt}\right) h_{nt}^{\alpha_0} l_{0nt}^{\alpha_2} + \alpha_1^{-1} \exp\left(x_{nt} B_1 + \epsilon_{1nt}\right) l_{1nt}^{\alpha_1} l_{0nt}^{\alpha_3} + \ldots$$

- The wage rate is the value of the marginal product for a standard labor unit times the efficiency rating of n:

$$w_{nt} \equiv w_t \exp\left(x_{nt} B_2 + \epsilon_{2nt}\right)$$

- Similarly:

$$r_{nt} \equiv r_t \exp\left(x_{nt} B_3 + \epsilon_{3nt}\right)$$

An International Comparison

- None of the specifications is rejected, all the coefficients are significant and are signed according to economic intuition.
- Contingent claims prices (inversely) track aggregate consumption quite well.
- We reject the null hypotheses that:
  - contingent claims prices between Germany and US are equal
  - contingent claims prices between different regions in the US are equal at the 0.05 but not at the 0.1 level
  - preferences between the two countries are the same
- With respect to purchasing power parity we:
  - do not reject the null that the value of marginal product of labor is equalized across both countries
  - reject the null that the premium to education is the same.

First Price Auctions (Guerre, Perrigne and Vuong, 2000)

Independent and identically distributed private values in a first price sealed bid auction

- One Thursday we will analyze other market structures.
- But for now consider a first price sealed bid (FPSB) auction for N players with independent and identically distributed (iid) private values (IPV):
  - each player $n \in \{1, \ldots, N\}$ simultaneously submits a bid denoted by $b_n \in \mathbf{R}^+$,
  - the player submitting the highest bid is awarded the (single) object up for auction
  - the winner pays what he or she bid.
  - the value of n winning the object is v_n where $v_n \in \mathbf{V}$ is iid drawn from $F(v)$ say.

- Let $W(b)$ denote the probability of winning the auction with bid b. That is:
$$W(b) \equiv \Pr\{b_k \leq b \text{ for all } k = 1, \ldots, N\}$$
- Then the maximization problem faced by player n can be written as:
$$\max_b (v_n - b) W(b)$$
- The first order condition (FOC) is:
$$(v_n - b_n) W'(b_n) - W(b_n) = 0 \qquad (4)$$
- The second order condition (SOC) of the optimization problem is:
$$\begin{aligned} 0 > SOC \equiv \frac{\partial}{\partial b} FOC &= \frac{\partial}{\partial b}\left[(v - b) W'(b) - W(b)\right] \\ &= (v - b) W''(b) - 2W'(b) \end{aligned}$$

Pure strategy best replies are increasing in valuations

- Totally differentiating the FOC with respect to b and v yields:

$$0 = W'(b_n)\, dv_n + \left[(v_n - b_n) W''(b_n) - 2W'(b_n) \right] db_n$$

and hence:

$$\frac{db_n}{dv_n} = \frac{-W'(b_n)}{(v_n - b_n) W''(b_n) - 2W'(b_n)} > 0$$

because $W'(b_n) > 0$ and the denominator of the quotient is the SOC.

- If the bidders are playing a pure strategy equilibrium with an interior solution, then b_n is increasing in v_n.

First Price Sealed Bid Auctions

Bayesian Nash Equilibrium with monotone bidding

- Consider Bayesian Nash Equilibrium (BNE) in which bidders follow a strategy $\beta : \mathbf{V} \rightarrow \mathbf{B} \equiv [0, \infty)$ where $\beta(v)$ is increasing in v.
- Then $\beta(v)$ has an inverse, which we denote by $\alpha : \mathbf{B} \rightarrow \mathbf{V}$ such that $\alpha[\beta(v)] = v$ for all v.
- Letting $G(b)$ denote the distribution of bids, it follows that:

$$W(b) \equiv \Pr\{b_k \leq b_n \text{ for all } k = 1, \ldots, N\} = G(b_n)^{N-1}$$

- From the monotonicity property of the BNE:

$$G(b) = F(\alpha(b))$$

First Price Sealed Bid Auctions

Identification when all bids are observed from the probability of winning

- Assume our data set consists of all the bids recorded in I auctions in which the same equilibrium is played.
- Let b_n^i for $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, I\}$ denote the bid by player n in the i^{th} auction.
- The probability of winning the auction, $W(b)$, and its derivative $W'(b)$ are identified.
- We rewrite the FOC, Equation (4) as:

$$v_n^i = b_n^i + \frac{W\left(b_n^i\right)}{W'\left(b_n^i\right)} \tag{5}$$

- This shows v_n^i is identified, and therefore so is $F(v)$.

- Alternatively note that the probability distribution of bids and its density, $G(b)$ and $G'(b)$, are identified.
- But the probability n wins with b_n is:

$$W(b_n) = G(b_n)^{N-1}$$

implying

$$W'(b_n) = (N-1)\, G(b_n)^{N-2} G'(b_n)$$

- We rewrite the FOC, Equation (4) as:

$$v_n^i = b_n^i + \frac{W(b_n^i)}{W'(b_n^i)} = b_n^i + \frac{G(b_n^i)}{(N-1)\, G'(b_n^i)} \tag{6}$$

- This shows v_n^i and hence $F(v)$ can also be directly identified off the bidding distribution $G(b)$.

First Price Sealed Bid Auctions

The distribution of winning bids

- Now suppose our data set consists of only the winning bid recorded in I auctions in which the same equilibrium is played.
- Let b^i for $i \in \{1, \ldots, I\}$ denote the winning bid in the i^{th} auction.
- Thus the distribution of winning bids, denoted by $H\left(b^i\right)$, is identified.
- Since the winning bid is defined as the highest one, $H(b)$ is just the probability that all the bids are less than b, implying:

$$H(b) = \Pr\left\{b_n^i \leq b \text{ for all } n = 1, \ldots, N\right\} = G(b)^N$$

- Consequently:

$$G(b) = H(b)^{\frac{1}{N}} \tag{7}$$

and

$$G'(b) = \frac{1}{N} H(b)^{\frac{1}{N}-1} H'(b) \tag{8}$$

- This shows the bidding distribution is identified from the data generating process of the winner's bid.

- Substituting Equations (7) and (8) back into Equation (6) gives:

$$v^i = b^i + \frac{G\left(b^i\right)}{\left(N-1\right)G'\left(b^i\right)} = b^i + \frac{NH\left(b\right)}{\left(N-1\right)H'\left(b\right)}$$

- This identifies the winning valuations, and hence their distribution, denoted by $F_W(v)$.
- But the distribution of the winning valuations is a one to one mapping of the distribution of all the valuations:

$$F_W(v) = \Pr\left\{v_n \leq v \text{ for all } n = 1, \ldots, N\right\} = F(v)^N$$

- Therefore $F(v)$ is identified off the winning bids alone using the equation:

$$F(v) = F_W(v)^{\frac{1}{N}}$$

- Now suppose as before:
  - each bidder knows her own valuation;
  - makes sealed bid (that is bids simultaneously).
- But instead of a FPSB auction, consider a SPSB auction, where the highest bidder wins the auction but only pays the second highest bid.
- Now it is a weakly dominant strategy for (each) n to bid her expected valuation, v_n.
- Intuitively, compared with bidding v_n:
  - bidding more implies winning some auctions that yield negative expected value, but leaves unchanged the expected value of any other auction that would be won;
  - bidding less implies losing some auctions that yield positive expected value, but leaves unchanged the expected value of any other auction that she would win.

Second Price Sealed Bid Auctions

Distribution of the second highest valuation

- Let $F(v)$ denote the distribution of valuations as before.
- Note first the obvious point that because players bid their valuations in SPSB auctions with private valuations, $F(v)$ is trivially identified if all the bids are observed.
- Now suppose only the winning price is observed.
- Then the probability distribution of the second highest valuation, which we now denote by $F_{N-1,N}(v)$, is identified.

- More generally, let $F_{i,N}(v)$ denote the distribution of the i^{th} order statistic.
- Then it can be shown (see Arnold, Balakrishnan and Nagaraja,1992 for example) that:

$$F_{i,N}(v) = \frac{N!}{(N-i)!\,(i-1)!} \int_{\underline{v}}^{F(v)} t^{i-1} (1-t)^{N-i}\, dt$$

- Note that $\underline{v}$ is identified (and a consistent estimate is the lowest winning bid observed in the data).
- Also:

$$\frac{\partial F_{i,N}(v)}{\partial v} = \frac{N!}{(N-i)!\,(i-1)!} F(v)^{i-1} \left[1 - F(v)\right]^{N-i}$$