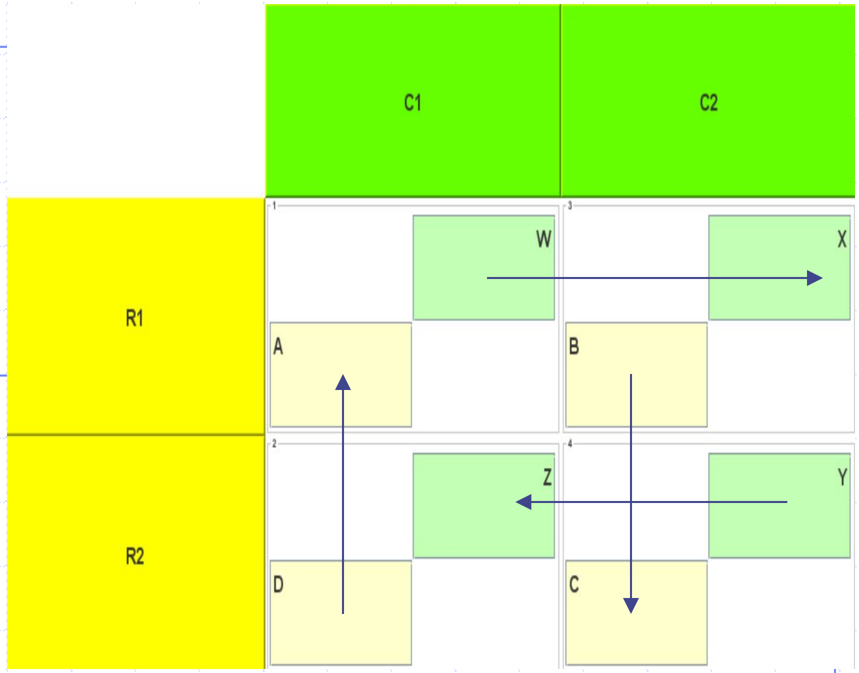
Session 5.2
The Existence of NE

At least one NE exists in every finite game. Here we formally covers the case where there are only two players, each with two strategies.

Why a pure strategy NE might not exist

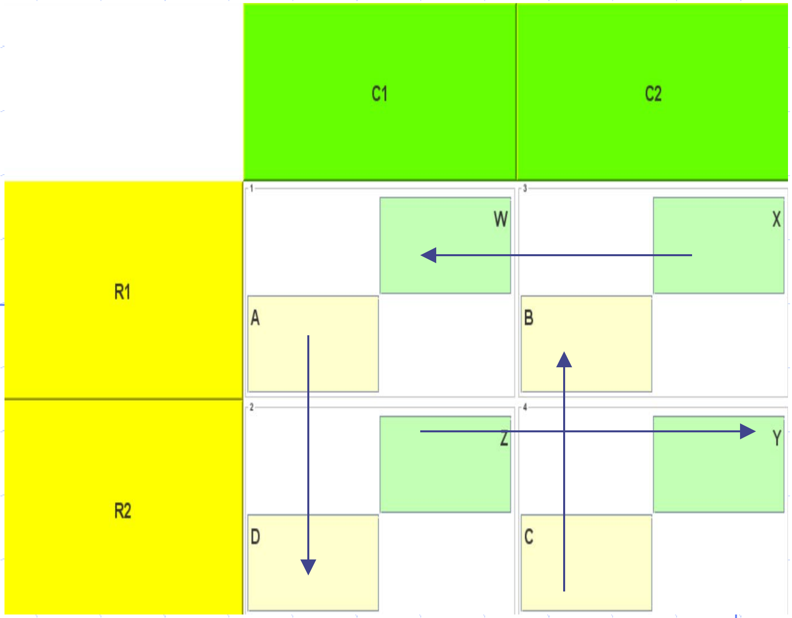
- The figure illustrates a game that does not have a pure strategy NE because:
  - D < A and B < C
  - W < X and Y < Z



- Notice the best response arrows form a clockwise circular path.

When a pure strategy NE does not exist

- Another possibility is that:
  - D > A and B > C
  - W > X and Y > Z

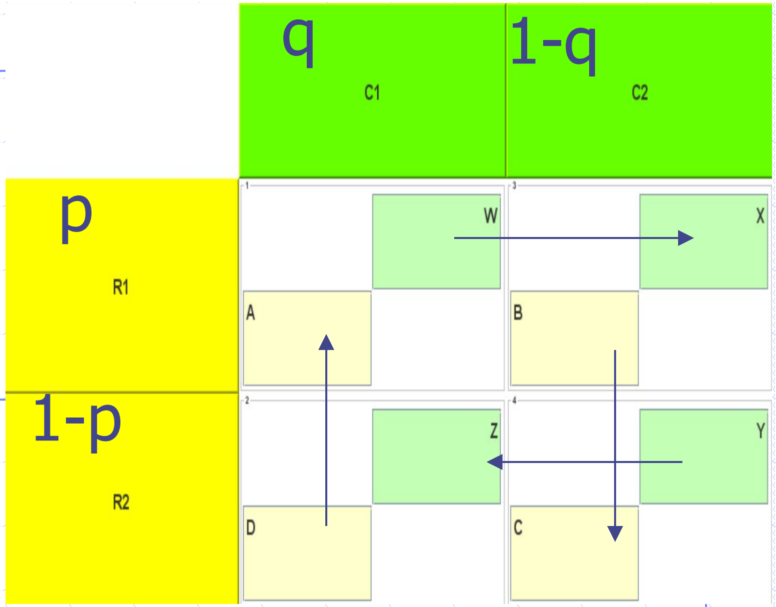


- Notice the best response arrows form a counterclockwise path.

- If neither of these two cases hold, at least one of the arrows must reverse direction, in the process revealing a pure strategy NE.

How to prove a mixed strategy NE exists

- Consider the clockwise case:
  - D < A and B < C
  - W < X and Y < Z
- We wish to show there are two probabilities p and q such that:
  - 0 < p < 1
  - 0 < q < 1



where yellow is indifferent between playing R1 and R2 while green is indifferent between playing C1 and C2. (If they strictly preferred one strategy over the other, then they would have picked a pure strategy, and we have already ruled that out.)

Simple algebra completes the proof

- If yellow is indifferent between R1 and R2 then:
  - $qA + (1 - q)B = qD + (1 - q)C$
- Solving for q:
  - $q(A - D + C - B) = C - B$
  - $q = (C - B)/(A - D + C - B)$
- Considering the first case:
  - A > D and C > B proving $0 < q < 1$.
- A similar argument shows that in the first case:
  - $0 < p < 1$ since $W < X$ and $Y < Z$
- The counterclockwise case is proved the same way.