Lecture 5
Strategic Uncertainty

A mixed strategy is a probability distribution over pure strategies. Every game has at least one pure and/or mixed strategy NE. Rational players using mixed strategies create strategic uncertainty.

Key words and phrases:

pure strategies, mixed strategies, existence of Nash equilibrium, strategic uncertainty

The empirical distribution

- Recall from the second lecture that the empirical distribution is the joint probability distribution of moves made by all the players.
- It is estimated from the relative frequencies of moves made at each information set.
- If you know the empirical distribution of everyone else, you can calculate the expected value from making your own choices.
- If everybody best replies to the empirical distribution, will we end up at a pure strategy Nash equilibrium?

Matching pennies

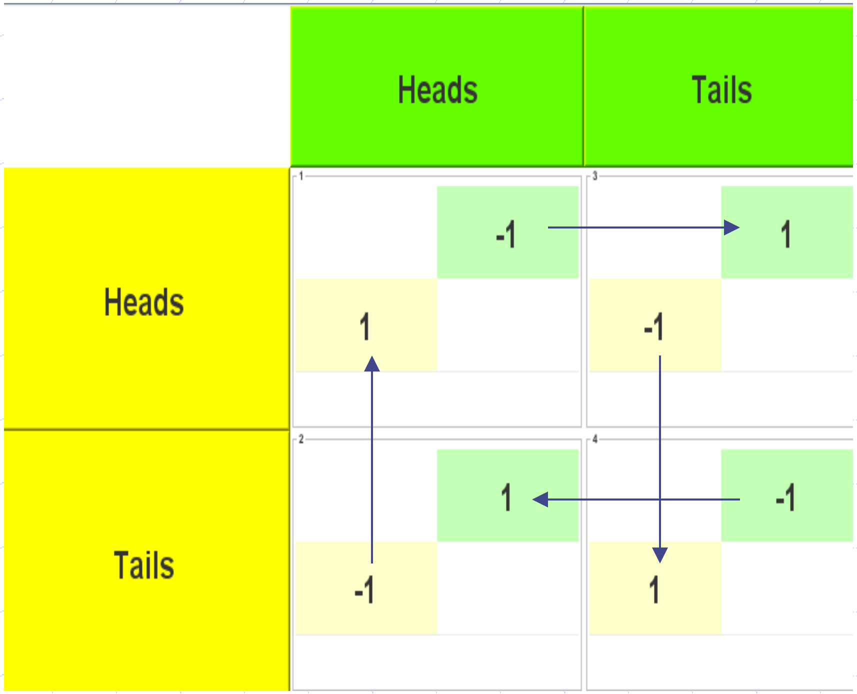
- In the zero sum game of Two-Up, a spinner tosses two coins (traditionally pennies) and the other player calls "evens" or "odds". The spinner wins if the other player calls incorrectly.

- In this equivalent game of Matching Pennies, yellow wins if the faces on the coins are the same (evens), while green wins if the faces are different (odds).

	Heads	Tails
Heads	-1 / 1	1 / -1
Tails	1 / -1	-1 / 1

The chain of best responses

- If yellow plays H, green's best reply is T, but if green plays T, then yellow's best reply is T too.
- Therefore (H,T) is not a Nash equilibrium.
- Using a similar argument, we can eliminate every strategy profile as being a pure strategy Nash equilibrium.



Avoiding losses in matching pennies

- If green plays H with probability less than 1/2, then yellow's expected gain from playing T is positive. In that case yellow's best reply is to play T all the time.
- Similarly yellow expects to gain from playing H if green plays H more than half the time: then yellow's best reply is to play H for sure.
- But if green randomly picks H with probability 1/2 each round, then yellow's expected profit, and hence green's, is zero regardless of yellow's strategy.
- Therefore, green expects to lose unless he independently mixes between heads and tails with probability one half.

Mixed strategy Nash equilibrium

- This simple example proves that not every game has a pure strategy Nash equilibrium.
- Suppose yellow selects H half the time. Then a best response of green is to select H half the time.
- Likewise suppose green selects H half the time. Then a best response of yellow is to select H half the time.
- Each player selecting H half the time is an example of a mixed strategy equilibrium.
- Since we have already confirmed no other equilibrium exists, this one is unique.

Mixed versus pure strategies

- When a player chooses randomly by following a probability distribution it is called a mixed strategy.
- This contrasts with a pure strategy, where a deterministic choice is made. A mixed strategy is a probability distribution over pure strategies.
- Suppose a player is indifferent between two (or more) pure strategies because they both (all) yield same expected payoff, given the strategies of the other players.
- Then the player would also be indifferent between all mixed strategies defined on the same support (of pure strategies).

Strategic uncertainty

- In the game of matching pennies all the uncertainty about the outcome arises solely from the strategies of the players.
- In general, strategic uncertainty arises when the players, not nature, through their own actions, induce the uncertainty.
- When there is strategic uncertainty NE:
  - does not predict a single outcome.
  - NE predicts the probability distribution of outcomes.

Mixed strategy NE occur frequently

- There are lots of examples of mixed strategy NE:
  - When traffic becomes congested on a freeway, the slowest lanes are the shortest and commuters are indifferent about which lane to select into. But length of time spent in the queue from selecting any given lane is random.
  - The outcomes of war are never perfectly foreseen. Military strategists give probabilities in which several strategies may look equally as plausible (leading to disagreement amongst the experts about the best one).
  - Street crime is randomly enforced, guided by probabilities of where police surveillance is most valuable, and offenders are often viewed as acting on impulse. In a mixed strategy equilibrium, they would be indifferent between breaking the law now versus waiting for the next opportunity.

The Ware case

- 10 years ago Ware received a patent for Dentosite that has since captured 60 percent share in the market. National had been the largest supplier of material for dental prosthetics before Dentosite was introduced.
- A new material FR 8420 was recently developed by NASA.
- If Ware develops a new composite with FR 8420 it will be a perfect substitute for Dentosite.
- If the technique is feasible then Ware would have just as good a chance as National of proving it first.
- If Ware develops it first, they could extend the patent protection to this technique and prevent any competitors.

Strategic considerations

- Ware's problem is bound to National's.
- Ware does not want to develop a technology that would not be used if the competitor does not develop it.
- If National develops the technology Ware cannot afford to drop out of the race.
- It all depends how people at National see this situation. Are Ware and National equally as well informed?

Some facts

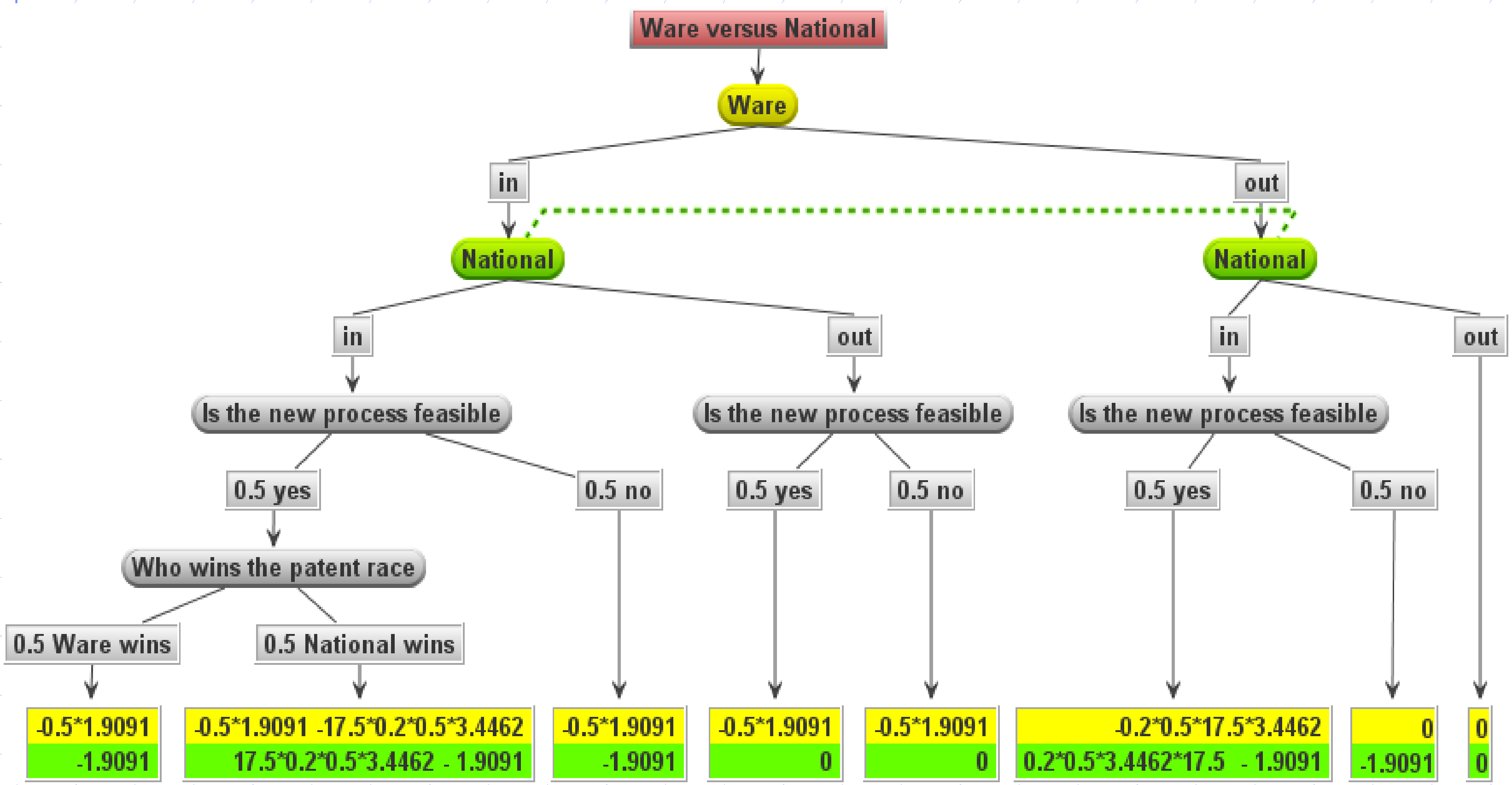
The Ware Case

10%	Discount/year
$1.9091	Value of $1 in years 1+2
$3.4462	Value of $1 in years 3+4+5+6+7

$0.500	Ware entry cost/yr	50%	Probability process feasible
$1.000	National entry cost/yr	50%	Ware chance of winning race
		50%	Ware market share if Nat'l enters mkt
$15	Range of possible	20%	Ware profit margin
$20	future annual sales (millions)		

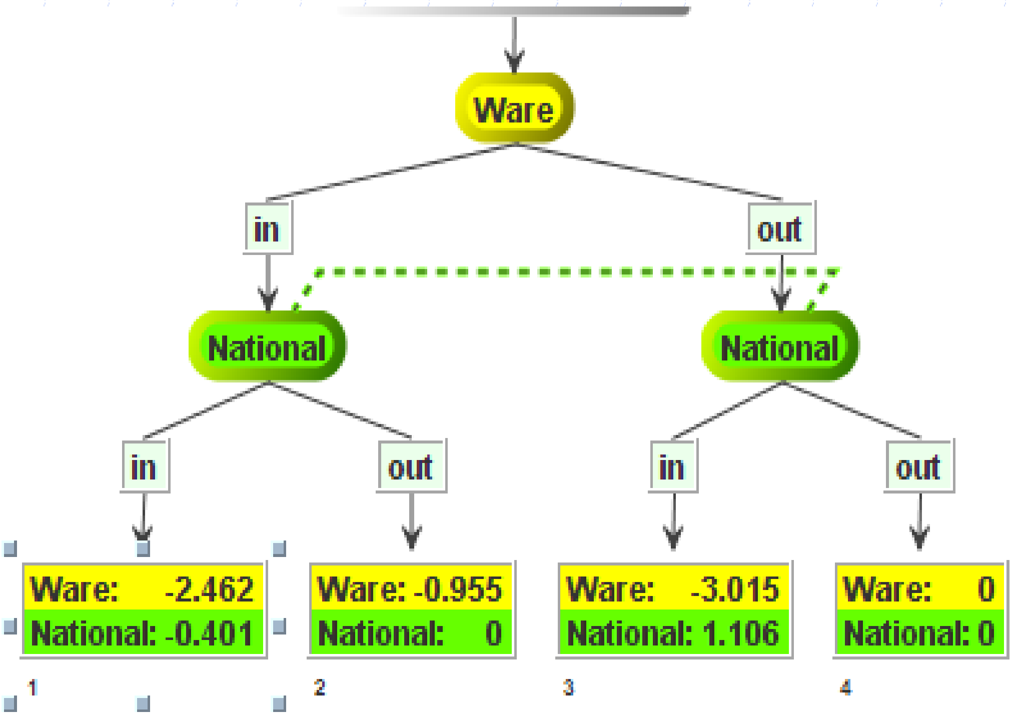
Ware case in the extensive form

- Using the facts, we can present the case as the following game tree:



Simplifying the extensive form

- Folding back the moves of chance that are related to developing a new technology we obtain the following simplification.



Strategic form

National Ware

	in	out
in	-0.4 -2.46	0 -0.95
out	1.11 -3.02	0 0

National's best reply to the empirical distribution of Ware

- National is indifferent between the two choices if the expected profits are equal.
- If Ware chooses "in" with probability p, the value to National from choosing "out" is 0, and the expected profits to National from choosing "in" are:

$$-0.401*p + 1.106*(1 - p)$$

- Solving for p we obtain:

$$-0.401*p + 1.106*(1 - p) = 0$$

$$=> \quad p = 0.734$$

- Thus if Ware enters with a higher probability than 0.734, then National should stay out, but if Ware enters with a lower probability than 0.734, National should enter itself.

Ware's best reply to the empirical distribution of National

- If National chooses "in" with probability q, then the expected value to Ware from choosing "in" is:

  $$-2.462*q - 0.955*(1 - q)$$

- Also, the expected value to Ware from choosing "out" is:

  $$-3.015*q$$

- Solving for q we obtain:

  $$2.462*q + 0.955*(1 - q) = 3.015*q$$

  $$\Rightarrow \quad q = .633$$

- Thus, if National enters with probability higher than 0.633, then Ware should enter too, but if National enters with probability lower than 0.633, then Ware should stay out.

Solution to the Ware case

- If Ware enters with probability p = 0.734, then a best response of National is to enter with probability q = 0.633.
- If National enters with probability q = 0.633, a best response of Ware is to enter with probability p = 0.734.
- Therefore the strategy profile p = 0.734 and q = 0.633 is a mixed strategy Nash equilibrium.
- In this case the equilibrium is the unique.

		National	
		"in" with probability q = 0.63	"out" with probability (1 - q) = 0.37
Ware	"in" with probability p = 0.73	pq = 0.46	p(1 - q) = 0.27
	"out" with probability (1 - p) = 0.27	(1 - p)q = 0.17	(1 - p)(1 - q) = 0.10

Mixed strategy NE defined

- For any finite game denote by:
  - s_{nj} the j^{th} pure strategy of player n where $1 \leqslant j \leqslant J$ and $1 \leqslant n \leqslant N$.
  - p_{nj} the probability that player n chooses pure strategy j.
  - $p_n = (p_{n1}, \ldots, p_{nJ})$ the vector of probabilities defining a mixed strategy for player n.
  - p_{-n} the mixed strategies of every player aside from n.
  - $p = (p_n, p_{-n})$ a mixed strategy profile for all the players.

- A mixed strategy NE is defined by a p as follows:

  For all n =1 , . . . , N, the expected payoff to n from playing p_n is at least as great as from playing any pure strategy s_{nj} when the other players select p_{-n}.

Existence of Nash equilibrium

- Consider any finite non-cooperative game, that is a game in extensive form with a finite number of nodes.
- If there is no pure strategy NE in the strategic form of the game, then there is a mixed strategy NE.
- In other words, every finite game has at least one NE in pure or mixed strategies.
- In Asymmetric Session 6.3 we prove this result for the special case in which there are two players, each with two pure strategies.

Strategic uncertainty

- Strategic uncertainty arises when the solution, is a Nash equilibrium with a mixed strategy.
- The uncertainty in equilibrium is directly attributable to the players’ choices rather than uncertainty about the environment.
- Finally, a mixed strategy equilibrium:
  - predicts the probabilities with which different courses of actions will be taken.
  - shows why in equilibrium the players are indifferent between those actions that have positive probability.