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Empirical Analysis of Limit Order Markets

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We provide empirical restrictions of a model of optimal order submissions in a limit order market. A trader's optimal order submission depends on the trader's valuation for the asset and the trade-offs between order prices, execution probabilities and picking off risks. The optimal order submission strategy is a monotone function of a trader's valuation for the asset. We test the monotonicity restriction in a sample of order submissions and their realized outcomes from the Stockholm Stock Exchange. We do not reject the monotonicity restriction for buy orders or sell orders considered separately, but reject the monotonicity restriction for buy and sell orders considered jointly.

1. INTRODUCTION

Many financial assets trade in limit order markets. In a limit order market, traders can submit market orders and limit orders. A market order fills immediately at the most attractive price posted by previously submitted limit orders in the limit order book. A limit order specifies a particular price, but does not guarantee that the order will be filled. Unfilled limit orders enter the limit order book, where they are stored until they are cancelled or filled by market orders.

A limit order offers the trader a better price than a market order, but there are costs to submitting a limit order. The limit order may fail to fill; we call the probability that the limit order fills the execution probability. The limit order may take time to fill. If the trader does not continuously monitor the limit order, then the limit order may fill when there is a change in the asset value; we call the expected loss from such fills the picking off risk. Many theoretical models of optimal order submissions are based on the trade-offs between order prices, execution probabilities and picking off risks. For example, in Cohen, Maier, Schwartz and Whitcomb (1981), Harris (1998), Parlour (1998) and Foucault (1999), market and limit order submitters consider the trade-offs. In Glosten (1994), Seppi (1997) and Biais, Martimort and Rochet (2000) limit order submitters consider the trade-offs.

Empirically, traders change their order submissions as market conditions change. Traders typically observe information on the number of unfilled limit orders in the book; as well as the bid–ask spread, equal to the difference between the prices of the lowest priced sell limit order and highest priced buy limit order in the book. Biais, Hillion and Spatt (1995) find that traders in the Paris Bourse are more likely to submit limit orders when the limit order book contains relatively few orders, or when the bid–ask spread widens. Similar results are found for other limit order markets: for example, Griffiths, Smith, Turnbull and White (2000) and Ranaldo (2003). If the execution probabilities for limit orders increase when there are fewer orders in the limit order book or when the bid–ask spread widens, the evidence may be consistent with the traders responding to changes in the trade-offs. Harris and Hasbrouck (1996) find that the expected pay-offs from limit orders relative to market orders increase when the bid–ask spread widens on the New York Stock Exchange. Traders change their order submission strategies as the bid–ask spread widens, tending to submit more limit orders. But are the traders' order submissions consistent with the theories based on the trade-offs?

We show how to determine if the traders' order submissions are consistent with theories based on the trade-offs. We provide empirical restrictions of a model of optimal order submissions based on the trade-offs, and develop a semiparametric test of them. We compute the test using the order flow and the limit order book for Ericsson, one of the most actively traded stocks on the Stockholm Stock Exchange.

A trader's optimal order submission in our model depends on the trader's valuation for the asset, and the trade-offs between the order price, the execution probability and the picking off risk of alternative order submissions. The optimal order submission strategy is a monotone step function of the trader's valuation, characterized in terms of threshold valuations. The threshold valuations are functions of the order prices and the trader's subjective beliefs about the execution probabilities and picking off risks. If the traders submit orders optimally according to our model, then the threshold valuations evaluated at the order prices, execution probabilities and picking off risks of the traders' actual order submissions form a monotone sequence. We use the actual order submissions, and the realized order fills to form estimates of execution probabilities and picking off risks. We use the estimates of the execution probabilities and picking off risks to form estimates of the threshold values at the actual order submissions, and test if they form a monotone sequence. The test does not require knowledge of the traders' valuations for the asset, or knowledge of the execution probabilities and picking off risks of orders not submitted by the traders.

A buy limit order fills after it becomes the highest-priced unfilled limit order in the book and a sell market order is submitted by another trader. Consequently, traders must predict future traders' order submissions to determine the execution probabilities and the picking off risks associated with alternative order submissions. In a stationary environment, the execution probabilities and picking off risks associated with alternative order submissions have sample analogues. Hotz and Miller (1993) and Manski (1993) suggest using non-parametric methods to estimate agents' conditional expectations to identify and estimate structural models. We follow such an approach, using non-parametric methods to estimate the execution probabilities and picking off risks for alternative order submissions. We condition on information in the limit order book and lagged trading activity to measure information available to the traders.

Our approach is related to work on structural estimation of auction models. The optimal bid in a first-price sealed bid auction with private values depends on the bidder's private value and the trade-off between the bid and the probability of the bid winning the auction. Guerre, Perrigne and Vuong (2000) provide necessary and sufficient conditions for the observed bids in a first-price sealed bid auction to be the Bayesian Nash equilibrium of an independent private values auction. One condition is that a function of the observed bids satisfies a monotonicity condition. Laffont and Vuong (1996) and Guerre *et al*. (2000) point out that the monotonicity condition can be used to test auction theory. The monotonicity restriction of the optimal order submission strategy that we test is a necessary condition for optimality of the observed order submissions given the actual dynamics of the limit order book.

In our sample, we do not reject the monotonicity restriction for buy or sell orders considered separately, but reject the monotonicity restriction for buy and sell orders considered jointly. The expected pay-offs from submitting limit orders with low execution probabilities are too low relative to the expected pay-offs from submitting limit orders with high execution probabilities to rationalize all the order submissions in our sample.

2. DESCRIPTION OF THE MARKET AND THE SAMPLE

In 1990 the Stockholm Stock Exchange completed the introduction of a limit order market system, the Stockholm Automated Exchange. Here, we briefly describe the Stockholm Automated Exchange and our sample. There are no floor traders, market makers, or specialists with unique quoting obligations or trading privileges. Trading is continuous from 10 a.m. to 2:30 p.m. with the opening price determined by a call auction. All order prices are required to be multiples of a fixed tick size. When prices are below 100 SKr, the tick size is 0·5 SKr and when prices are above 100 SKr, the tick size is 1 SKr. During the sample period, $1 was approximately equal to 6·25 SKr. The order quantity must be a multiple of a round lot, with a typical round lot quantity of 100 shares.

All trading is between market and limit orders. Unfilled limit orders are stored in the electronic limit order book and are automatically filled by market orders. Unfilled limit orders in the order book are prioritized first by price and then by time of submission. The prices of the sell limit orders in the book are called ask quotes and the prices of the buy limit orders in the book are called bid quotes. If a market order is for a smaller quantity than the quantity at the best quote in the limit order book, the market order will completely fill at a price equal to the best quote. If a market order cannot be filled completely at the best quote, it will transact with multiple quotes in the book until either it is completely filled or the book is empty. Any unfilled portion of a market order converts into a limit order.

A limit order can be cancelled at any time at no cost. Traders can also submit hidden limit orders, where only a portion of the order quantity is displayed in the order book. The hidden part of a limit order has lower priority than all displayed limit orders at the same order price level.

Orders can only be directly submitted to the trading system by exchange member firms. A member firm submits orders as a broker for its customers and as a dealer for itself. During our sample period, there were 24 exchange member firms. We refer to the member firms as brokers. The brokers are directly connected to the trading system and observe all price quotes with the corresponding total order quantities in the limit order book. The brokers' information is updated almost instantaneously after order submissions or cancellations. Traders who are not directly connected to the system can obtain information about the five best bid and ask quotes and the corresponding order quantities in the limit order book through information vendors such as Reuters or Telerate.

The Stockholm Stock Exchange was the only authorized marketplace for equity trading in Sweden until 1 January, 1993. But many of the stocks listed on the exchange were also cross-listed on foreign exchanges; trading in London on the international stock exchange automated quotation (SEAQ) system and in the U.S. on the national association of securities dealers automated quotation (NASDAQ) system accounted for a significant fraction of the trading of many Swedish stocks.

Brokers can settle trades larger than 100 round lots outside the Stockholm Automated Exchange system. An internal cross is a trade of 100–500 round lots where the broker represents both sides of the trade. A block trade is a trade greater than 500 round lots.

We obtained the order records and the trade records directly from the Stockholm Automated Exchange system for the 59 trading days between 3 December, 1991 and 2 March, 1992 for Ericsson. The order records is a chronological list of order submissions, changes in the outstanding order quantities, and order cancellations. The trade records is a chronological list of transactions. Each limit order receives a unique code, and subsequent changes in the outstanding order quantity are recorded using the same code. We combine changes in the outstanding order quantity and the transactions to determine whether a change in the order quantity was caused by a trade or a cancellation. We reconstruct complete transaction and cancellation histories for limit orders and the entire history of the order book over our sample.

TABLE 1

Daily trading activity

	Mean	Std dev.	Min.	Max.
Daily closing mid-quote (SKr)	110·15	9·19	89·00	127·00
Daily close-to-close return (%)	0·21	3·04	−6·31	11·44
Daily close-to-close return (%) on NASDAQ, 2 January, 1989–31 December, 1993	0·10	2·45	−16·95	13·69
Number of active brokers per day	19·28	2·28	14·00	23·00
Daily trading volume in millions of SKr				
Stockholm Automated Exchange	38·77	19·53	11·58	114·88
Internal crosses	12·20	9·04	1·48	57·51
Block trades	0·39	0·92	0·00	5·45
After-hours trades	4·66	6·15	0·00	28·26
Total trading volume	56·02	29·52	13·06	201·28
Daily number of Stockholm Automated Exchange orders, 10:03 a.m.–2:30 p.m.				
All orders	364·23	141·13	128	733
Limit orders	212·18	77·58	73	408
Market orders	152·04	67·14	55	330

The table reports summary statistics on the daily trading activity of Ericsson. The daily close-to-close returns are calculated using the mid-quotes. The daily close-to-close returns for Ericsson shares traded on NASDAQ are calculated using daily data from the Center for Research in Security Prices. The number of active brokers per day is defined as the number of brokers who made at least one trade on a given trading day.

We have detailed information, but there are limitations. We only identify the broker submitting the order; we cannot separately identify the orders that a broker submits for his customers from the orders that a broker submits for himself. We do not observe whether or not an order includes a hidden order quantity component. We only infer that an order must have involved a hidden portion if the displayed portion of the hidden order is executed in full. In our sample, there are few hidden orders whose displayed portions do fully execute. The first limitation causes us to focus on how a representative trader decides on his order submissions.

Some limit orders remain in the order book at the end of our sample period. To minimize the effects of the resulting censoring bias on our empirical work, we do not use orders submitted during the last two days of our sample. Only 2·8% of the orders remain in the system for more than two trading days, and 62·3% of such orders are eventually cancelled. We discard orders submitted during the first 3 minutes of the trading day to ensure that the sample reflects only continuous trading. The filtering rules leave us with 20,760 observations of individual order submissions and their realized fills.

Table 1 reports descriptive statistics on the daily trading activity for Ericsson. The tick size varies between 0·5 SKr and 1 SKr since the price is both less than 100 SKr and greater than 100 SKr in our sample. The mean daily close-to-close return is 0·21% with a standard deviation of 3·04%. For comparison, the table also reports statistics on close-to-close returns computed using prices of Ericsson shares on NASDAQ from 2 January, 1989, through to 31 December, 1993, using data from the Center for Research in Security Prices. The return distribution in our sample is not unusual.

The fourth row of Table 1 reports information on the daily number of active brokers in Ericsson. All 24 brokers are active in Ericsson over the sample period. Sorting the brokers by their share of trading volume, the top three brokers each transact 10% to 11%, and the next seven brokers each transact 5% to 9%. The shares are almost identical for order submissions. No single broker dominates the trading of Ericsson.

TABLE 2

Order submissions

Order	Number of orders	Execution probability Mean	Time to execution Mean	Order quantity Mean	Std dev.
		Buy orders			
Market	6031	1·00	0·00	19·72	39·21
1 Tick limit	4225	0·68	24·55	25·14	37·60
2 Tick limit	992	0·33	73·94	29·63	44·58
3 Tick limit	893	0·12	172·33	14·65	32·38
		Sell orders			
Market	4044	1·00	0·00	30·73	48·90
1 Tick limit	3212	0·63	18·31	36·08	49·48
2 Tick limit	800	0·28	84·21	37·08	44·87
3 Tick limit	563	0·13	170·95	23·73	91·29

The table reports descriptive statistics for the order submissions in Ericsson. The execution probability is defined as the fraction of the order quantity that fills within two trading days of the order submission. The time to execution is the number of minutes elapsed from the order submission until the order executes. When there are multiple fills we compute the time to execution by weighing each fill according to the fraction of the order quantity that is filled. Fills that occur later than two trading days after the order was submitted are ignored. The order quantity is measured in 100's of shares. There are 20,760 orders.

The fifth row of Table 1 reports the daily trading volume on the Stockholm Automated Exchange. The sixth through eighth rows of Table 1 report descriptive statistics for orders crossed internally by brokers, block trades during regular trading hours, and after-hours trading. The ninth row reports the total trading volume. In our subsequent analysis, we focus on traders' order submissions within the Stockholm Automated Exchange and abstract from traders' decisions to use the automated system itself or not.

The first column of Table 2 reports the number of buy and sell market and limit orders. The second column reports the mean execution probabilities, equal to the mean of the fraction of the limit order quantity that is filled within two trading days of order submission. The execution probabilities show the unconditional trade-off between the execution probability and the order price; limit orders at prices farther away from the quotes have lower execution probabilities than limit orders with prices closer to the quotes. The third column of Table 2 reports the mean time to execution for limit orders. Limit orders at prices farther away from the quotes take longer to execute than limit orders at prices closer to the quotes. The final three columns of Table 2 report the mean and standard deviation of the order quantity.

The top plot in Figure 1 is the sample survivor function for limit orders. The survivor function evaluated at t is the probability that a limit order remains outstanding for at least t minutes. If a limit order is completely filled at time t, the order's execution time is t. If a limit order is partially filled at time t, we weight the fill time by the fraction of the submitted order quantity filled to compute the order's execution time. Subsequent order fills or cancellations are weighted in the same manner. Most limit orders leave the book quickly; only 4·6% of the limit orders last for more than one trading day (270 min). The bottom two plots in Figure 1 show the cumulative distribution function for order execution and cancellation times. Although the cumulative distributions for execution and cancellation times are different, most executions or cancellations do occur within 3 h (180 min) of the order submission.

The first six rows of Table 3 report information about the order quantities in the limit order books. The average quantity at the best bid or ask quote is roughly nine times the average market order quantity; only 12 market orders in our sample are for quantities that are larger than the quantities available in the order book at the best quote at the time of order submission. The order

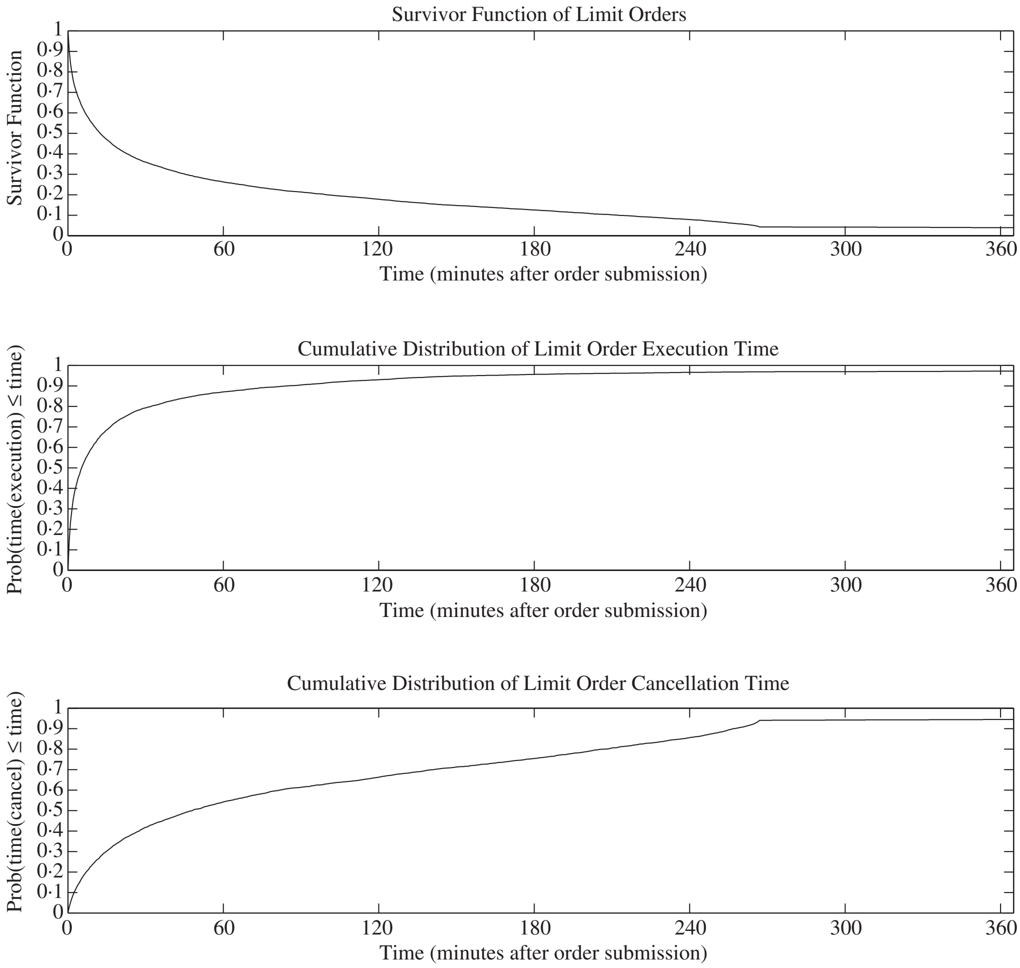

FIGURE 1

Limit order execution and cancellation times.

The top graph plots the survivor function for limit orders. The survivor function at any time is defined as the probability that a limit order is outstanding a given number of minutes after it was submitted. The middle and the bottom graphs plot the cumulative distribution functions for the limit order execution and cancellation times. All three functions are computed for orders submitted between 10:03 a.m. and 2:30 p.m. There are a total of 11,760 limit orders submitted. The survivor and distribution functions are calculated by assigning a weight to each observation equal to the fraction of the order quantity filled or cancelled. Limit orders submitted during the last two trading days in our sample are not used in the calculations

quantities in the limit order book are volatile. The standard deviations of the order quantities at the three best bid or ask quote levels are all greater than 173 round lots. The standard deviations of the cumulative order quantities are all greater than 294 round lots.

The last six rows of Table 3 provide information on the bid–ask spread and the distances between price quotes in the book. The bid–ask spread typically is one tick, and relatively constant over our sample. The distances between other price quotes in the book behave similarly.

On average, there is a trade-off between order price, execution probability and the length of time that an order remains unfilled in the limit order book. While most orders are either executed

TABLE 3

Order books

	Mean	Std dev.	Min.	Max.
Order quantities at quotes (100's of shares)				
3rd Ask	169·8	175·9	1·0	1061·1
2nd Ask	260·8	203·0	1·0	1161·0
1st Ask	200·9	194·1	1·0	1314·0
1st Bid	185·9	173·8	1·0	1504·2
2nd Bid	242·9	217·7	1·0	1504·2
3rd Bid	167·9	190·9	1·0	1355·2
Cumulative order quantities (100's of shares)				
1st + 2nd + 3rd Ask	631·5	368·8	13·0	1935·6
1st + 2nd Ask	461·7	294·9	2·0	1809·0
1st + 2nd Bid	428·8	310·2	5·0	2176·9
1st + 2nd + 3rd Bid	596·8	396·0	15·0	2730·4
Distance between price quotes (ticks)				
Bid–ask spread	1·1	0·3	1·0	6·0
3rd Ask–2nd ask	1·1	0·4	1·0	6·0
2nd Ask–1st ask	1·1	0·2	1·0	6·0
1st Bid–2nd bid	1·1	0·2	1·0	8·0
2nd Bid–3rd bid	1·1	0·5	1·0	10·0

Descriptive statistics for the order books. The statistics in the table are computed for each order book observed in the market immediately prior to an order submission. There are 20,760 observations.

or cancelled quickly, the mean time to execution is more than an hour for two and three tick limit orders. When time elapses between order submission and execution, the order may fill when there is a change in the asset's value; limit orders may face picking off risk. Finally, the number of orders in the limit order book itself is volatile—observing such information may help a trader predict execution probabilities and picking off risks. In the next section, we model a representative trader's optimal order submission strategy in such a market.

3. MODEL

3.1. *Assumptions*

At time t, one trader has the opportunity to submit an order. The trader is risk neutral, characterized by his valuation for the asset, v_t, and the quantity that he wishes to trade, q_t. We decompose v_t into two components,

$$v_t = y_t + u_t. \tag{1}$$

The random variable y_t is the common value of the asset at t; one interpretation is that the common value is equal to the traders' expectations of the liquidation value of the asset. The common value changes as the traders learn new information, with

$$y_{t+1} = y_t + \delta_{t+1}. \tag{2}$$

Innovations in the common value δ_{t+1} satisfy

$$E_t[\delta_{t+1}] = 0, \tag{3}$$

where the subscript t denotes conditioning on information available after the common value is known at t, but before the trader at t arrives. The distribution of common value innovations has bounded support. Common value innovations are drawn from a stationary process, and the distribution of the innovations is conditioned on the history of common value innovations through a finite-dimensional vector of sufficient statistics. Conditioning on a set of sufficient statistics allows for persistence in the higher-order moments of the innovations. For example, current common value volatility may depend on lagged common value volatility.

The random variable u_t is the trader's private value for the asset. Different traders can have different private values—heterogeneity in the traders' private values for the asset creates potential gains from trade. Traders may have different private values as a result of endowment shocks or differences in their current portfolios. The private value is drawn from a continuous distribution

$$\Pr_t(u_t \leq u) \equiv G_t(u), \tag{4}$$

where, as above, the subscript t denotes conditioning on information available after the common value is known at t, but before the trader at t arrives. The distribution of the private value has bounded support. The private value is drawn from a stationary process and the conditional distribution of the private value depends on the same finite-dimensional vector of sufficient statistics as the conditional distribution of common value innovations. For example, the current distribution of the private values may depend on lagged common value volatility.

Once a trader arrives at the market, his private value remains fixed while he has an order outstanding. At a random time, $t + \tau_{\text{cancel}}$, after the trader arrives at the market, the pay-off from any unfilled limit orders submitted by the trader will go to zero, causing the trader to cancel any unfilled limit orders. The trader does not know the cancellation time at time t. The conditional distribution of the cancellation time depends on the same finite-dimensional vector of sufficient statistics as the common value innovations. For example, the distribution of the cancellation time may depend on lagged common value volatility. Let $\Upsilon < \infty$ be the maximum possible lifetime of the order,

$$\Pr_t(\tau_{\text{cancel}} \leq \Upsilon < \infty) = 1. \tag{5}$$

The assumptions of bounded common value innovations, bounded private values and bounded cancellation times are used to show that orders in the limit order book are a bounded distance away from the common value.

The trader's desired order quantity, q_t, is independent of the trader's valuation, and is drawn from a distribution with bounded support. The conditional distribution of the order quantity depends on the same set of sufficient statistics as the common value innovations. For example, the distribution of order quantity may depend on lagged common value volatility.

At t, the trader has a single opportunity to submit either a market order or a limit order. The trader observes the current limit order book, the current common value, and the history of common value innovations. The trader pays a cost of c per share to submit an order. The cost is the same for all types of orders submitted.

The decision indicators $d^{\text{sell}}_{t,s} \in \{0, 1\}$ for $s = 0, 1, \ldots, S$ and $d^{\text{buy}}_{t,b} \in \{0, 1\}$ for $b = 0, 1, \ldots, B$ denote the trader's order submission at t. The trader chooses an order submission from a finite set: $S < \infty$ and $B < \infty$. If the trader submits a sell market order, the order price is the best bid quote and $d^{\text{sell}}_{t,0} = 1$. If the trader submits a sell limit order at the price s ticks above the current best bid quote, $d^{\text{sell}}_{t,s} = 1$. If the trader submits a buy market order, the order price is the best ask quote and $d^{\text{buy}}_{t,0} = 1$. If the trader submits a buy limit order at the price b ticks below the current best ask quote, $d^{\text{buy}}_{t,b} = 1$. If the trader does not submit any order at time t, $d^{\text{sell}}_{t,s} = 0$ for all s and $d^{\text{buy}}_{t,b} = 0$ for all b.

Suppose a trader with valuation $v_t = y_t + u_t$ at t submits a buy order of quantity q_t at a price $p^{\text{buy}}_{t,b}$, b ticks below the current best ask quote: $d^{\text{buy}}_{t,b} = 1$. Define $dQ_{t,t+\tau}$ as the number of shares of the order submitted at t that transact at $t + \tau$. If the realized cancellation time is

$t + \tau_{\text{cancel}}$, $dQ_{t,t+\tau} = 0$ for all $\tau \geq \tau_{\text{cancel}}$. If the trader submits a limit order, he will not know if or when the order will execute in the future: $dQ_{t,t+\tau}$ for $\tau > 0$ is a random variable. For any possible realization of the future order flow, the future common value and the cancellation time, the total quantity of the order that eventually transacts must be less than the order quantity

$$\sum\nolimits_{\tau=1}^{\Upsilon} dQ_{t,t+\tau} \leq q_t. \tag{6}$$

The pay-off that the trader receives from transacting $dQ_{t,t+\tau}$ at time $t + \tau$ at price $p_{t,b}^{\text{buy}}$ is

$$dQ_{t,t+\tau}(y_{t+\tau} + u_t - p_{t,b}^{\text{buy}}) = dQ_{t,t+\tau}(v_t - p_{t,b}^{\text{buy}}) + dQ_{t,t+\tau}(y_{t+\tau} - y_t), \tag{7}$$

where $y_{t+\tau}$ is the common value at $t + \tau$. The term $dQ_{t,t+\tau}(v_t - p_{t,b}^{\text{buy}})$ is the pay-off that a transaction of $dQ_{t,t+\tau}$ would earn upon immediate execution at price $p_{t,b}^{\text{buy}}$. The term $dQ_{t,t+\tau}(y_{t+\tau} - y_t)$ is the number of shares transacted in τ periods multiplied by the change in the common value. Summing over all possible transaction times for the order and including the cost of submitting the order, the realized pay-off is

$$U_{t,t+\Upsilon} = \sum\nolimits_{\tau=0}^{\Upsilon} dQ_{t,t+\tau}(v_t - p_{t,b}^{\text{buy}}) + \sum\nolimits_{\tau=0}^{\Upsilon} dQ_{t,t+\tau}(y_{t+\tau} - y_t) - q_t c. \tag{8}$$

If the order does not execute, $dQ_{t,t+\tau} = 0$ for $0 \leq \tau \leq \Upsilon$, and the trader's realized pay-off is $-q_t c$.

We define the execution probability as

$$\psi_t^{\text{buy}}(b, q_t) \equiv E_t\left[\sum\nolimits_{\tau=0}^{\Upsilon} \frac{dQ_{t,t+\tau}}{q_t} \middle| d_{t,b}^{\text{buy}} = 1, q_t\right] \tag{9}$$

and the picking off risk as

$$\xi_t^{\text{buy}}(b, q_t) \equiv E_t\left[\sum\nolimits_{\tau=0}^{\Upsilon} \frac{dQ_{t,t+\tau}}{q_t}(y_{t+\tau} - y_t) \middle| d_{t,b}^{\text{buy}} = 1, q_t\right]. \tag{10}$$

The conditional expectations in equations (9) and (10) do not depend on the trader's private value. If the order is a market order, the execution probability is one and the picking off risk is zero. In taking the expectations in equations (9) and (10), the trader is accounting for the different possible realizations of other traders' future order submissions, future common values and cancellation times.

The trader's expected pay-off is the expected value of equation (8), conditional on the trader's information set, which includes the current limit order book; the current common value; the history of common value innovations; the trader's private value; the trader's order quantity; and the trader's order submission:

$$E_t[U_{t,t+\Upsilon} \mid d_{t,b}^{\text{buy}} = 1, u_t, q_t] = q_t \psi_t^{\text{buy}}(b, q_t)(v_t - p_{t,b}^{\text{buy}}) + q_t \xi_t^{\text{buy}}(b, q_t) - q_t c. \tag{11}$$

The first term in the trader's expected pay-off is the expected number of shares that will eventually transact multiplied by the pay-off per share for an immediate transaction at price $p_{t,b}^{\text{buy}}$. The second term in the trader's expected pay-off is the covariance of changes in the common value with the quantity of the order that transacts. The final term in the trader's expected pay-off is the cost of submitting the order. The expected pay-off to a trader submitting a sell order for q_t shares at a price s ticks above the current best bid quote is defined similarly.

The trader submits the order that maximizes his expected pay-off, conditional on his information, private value, and order quantity q_t,

$$\max_{\{d^{\text{sell}}_{t,s}\}_{s=0}^{S},\{d^{\text{buy}}_{t,b}\}_{b=0}^{B}} \sum\nolimits_{s=0}^{S} d^{\text{sell}}_{t,s} E_t[U_{t,t+\Upsilon} \mid d^{\text{sell}}_{t,s} = 1, u_t, q_t] + \sum\nolimits_{b=0}^{B} d^{\text{buy}}_{t,b} E_t[U_{t,t+\Upsilon} \mid d^{\text{buy}}_{t,b} = 1, u_t, q_t], \tag{12}$$

subject to

$$d^{\text{sell}}_{t,s} \in \{0,1\}, \text{ for } s = 0,1,\ldots,S, \; d^{\text{buy}}_{t,b} \in \{0,1\}, \text{ for } b = 0,1,\ldots,B, \tag{13}$$

$$\sum\nolimits_{s=0}^{S} d^{\text{sell}}_{t,s} + \sum\nolimits_{b=0}^{B} d^{\text{buy}}_{t,b} \leq 1. \tag{14}$$

Equation (14) is the constraint that at most one order is submitted. Let $d^{\text{sell}*}_t(s, u_t, q_t)$ and $d^{\text{buy}*}_t(b, u_t, q_t)$ be the optimal strategy, detailing the trader's submission as a function of his beliefs and information, private value and order quantity.

Foucault (1999) solves for the unique stationary equilibrium in a model satisfying our assumptions. In Foucault (1999), common value innovations are independent and identically distributed binomial random variables; traders' private values are independent and identically distributed binomial random variables; limit orders last at most one period; the traders' desired order quantity is one unit; and the order submission cost is zero. Our model allows for more general distributions for the common value innovations, the private values, cancellation times and desired order quantity, and allows for positive order entry costs. Our characterization of the optimal order submission strategy provides a necessary condition for a Nash equilibrium in our model.

3.2. *Optimal order submission strategy*

Lemma 1. *Suppose that a trader with private value u and quantity q optimally submits a buy order at price $b \geq 0$ ticks below the ask quote: $d^{\text{buy}*}_t(b, u, q) = 1$.*

(1) *A trader with private value $u' > u$ and quantity q submits a buy order at a price b' ticks below the ask quote: $d^{\text{buy}*}_t(b', u', q) = 1$, with the execution probability higher at b' than at b,*

$$\psi^{\text{buy}}_t(b', q) \geq \psi^{\text{buy}}_t(b, q). \tag{15}$$

(2) *Suppose that the execution probabilities are strictly decreasing in the distance between the limit order price and the best ask quote, $\psi^{\text{buy}}_t(j+1, q) < \psi^{\text{buy}}_t(j, q)$, for all $j = 0, 1, \ldots, B-1$. Then a trader with private value $u' > u$ for q shares submits a buy order at a price b' ticks below the ask quote: $d^{\text{buy}*}_t(b', u', q) = 1$, with $b' \leq b$.*

Analogous results hold on the sell side.

The optimal order submission depends on the trader's valuation. The common value is fixed at t so that the only source of heterogeneity in the decision at t is the trader's private value. If the trader buys, the higher the trader's private value, the higher the execution probability is for the trader's optimal buy order. If the trader sells, the lower the trader's private value, the higher the execution probability is for the trader's optimal sell order.

Lemma 1 and the discrete price grid imply that we can partition the set of valuations into intervals. All traders wishing to trade the same quantity whose valuations lie within the same interval submit the same order. Define the threshold valuation $\theta_t^{\text{buy}}(b, b', q)$ as the valuation of a trader who is indifferent between submitting a buy order at price $p_{t,b}^{\text{buy}}$ and a buy order at price $p_{t,b'}^{\text{buy}}$,

$$\theta_t^{\text{buy}}(b, b', q) = p_{t,b}^{\text{buy}} + \frac{\left(p_{t,b}^{\text{buy}} - p_{t,b'}^{\text{buy}}\right)\psi_t^{\text{buy}}(b', q) + \left(\xi_t^{\text{buy}}(b', q) - \xi_t^{\text{buy}}(b, q)\right)}{\psi_t^{\text{buy}}(b, q) - \psi_t^{\text{buy}}(b', q)}. \tag{16}$$

The threshold valuation for a buy order at price $p_{t,b}^{\text{buy}}$ and not submitting an order is

$$\theta_t^{\text{buy}}(b, \text{NO}, q) = p_{t,b}^{\text{buy}} + \frac{-\xi_t^{\text{buy}}(b, q) + c}{\psi_t^{\text{buy}}(b, q)}. \tag{17}$$

The threshold valuation for a sell order at price $p_{t,s}^{\text{sell}}$ and a sell order price at $p_{t,s'}^{\text{sell}}$ is

$$\theta_t^{\text{sell}}(s, s', q) = p_{t,s}^{\text{sell}} - \frac{\left(p_{t,s'}^{\text{sell}} - p_{t,s}^{\text{sell}}\right)\psi_t^{\text{sell}}(s', q) + \left(\xi_t^{\text{sell}}(s, q) - \xi_t^{\text{sell}}(s', q)\right)}{\psi_t^{\text{sell}}(s, q) - \psi_t^{\text{sell}}(s', q)}. \tag{18}$$

The threshold valuation for a limit sell order at price $p_{t,s}^{\text{sell}}$ and not submitting an order is

$$\theta_t^{\text{sell}}(s, \text{NO}, q) = p_{t,s}^{\text{sell}} - \frac{\xi_t^{\text{sell}}(s, q) + c}{\psi_t^{\text{sell}}(s, q)}. \tag{19}$$

The threshold valuation for a sell order at price $p_{t,s}^{\text{sell}}$ and a buy order at price $p_{t,b}^{\text{buy}}$ is

$$\theta_t(s, b, q) = p_{t,s}^{\text{sell}} + \frac{\left(p_{t,b}^{\text{buy}} - p_{t,s}^{\text{sell}}\right)\psi_t^{\text{buy}}(b, q) - \left(\xi_t^{\text{sell}}(s, q) + \xi_t^{\text{buy}}(b, q)\right)}{\psi_t^{\text{sell}}(s, q) + \psi_t^{\text{buy}}(b, q)}. \tag{20}$$

Let $\mathcal{B}_t^*(q)$ index the set of buy order prices that are optimal for some trader who wishes to trade q shares at time t,

$$\mathcal{B}_t^*(q) \equiv \left\{b \;\middle|\; d_t^{\text{buy}*}(b, u, q) = 1 \text{ for some } u\right\}, \tag{21}$$

with elements $b_{i,t}^*(q)$, for $i = 1, \dots, I$, ordered by the execution probabilities,

$$\psi_t^{\text{buy}}(b_{i,t}^*(q), q) > \psi_t^{\text{buy}}(b_{i+1,t}^*(q), q). \tag{22}$$

Here, $b_{1,t}^*(q)$ indexes the buy order with the highest execution probability that some trader would submit, and $b_{I,t}^*(q)$ indexes the buy order with lowest execution probability that some trader would submit. Typically $b_{1,t}^*(q) = 0$; a market buy order is an optimal buy order submission for some trader. Define $\mathcal{S}_t^*(q)$ similarly.

Lemma 2.

$$\theta_t^{\text{buy}}(b_{1,t}^*(q), b_{2,t}^*(q), q) > \theta_t^{\text{buy}}(b_{2,t}^*(q), b_{3,t}^*(q), q) > \cdots > \theta_t^{\text{buy}}(b_{I-1,t}^*(q), b_{I,t}^*(q), q), \tag{23}$$

$$\theta_t^{\text{sell}}(s_{J-1,t}^*(q), s_{J,t}^*(q), q) > \theta_t^{\text{sell}}(s_{J-2,t}^*(q), s_{J-1,t}^*(q), q) > \cdots > \theta_t^{\text{sell}}(s_{1,t}^*(q), s_{2,t}^*(q), q), \tag{24}$$

$$\theta_t^{\text{buy}}(b_{I-1,t}^*(q), b_{I,t}^*(q), q) > \theta_t(s_{J,t}^*(q), b_{I,t}^*(q), q) > \theta_t^{\text{sell}}(s_{J-1,t}^*(q), s_{J,t}^*(q), q). \tag{25}$$

To describe the optimal decision rule, define the marginal thresholds for sellers and buyers as

$$\begin{aligned}\theta_t^{\text{buy}}(\text{Marginal}_t(q), q) &= \max(\theta_t(s^*_{J,t}(q), b^*_{I,t}(q), q), \theta_t^{\text{buy}}(b^*_{I,t}(q), \text{NO}, q)),\\ \theta_t^{\text{sell}}(\text{Marginal}_t(q), q) &= \min(\theta_t(s^*_{J,t}(q), b^*_{I,t}(q), q), \theta_t^{\text{sell}}(s^*_{J,t}(q), \text{NO}, q)).\end{aligned} \tag{26}$$

If the buyer and seller marginal thresholds are equal to each other, all traders find it optimal to submit an order. Otherwise, some traders find it optimal not to submit any order.

Lemma 3. *The optimal order submission strategy is*

$$d_t^{\text{buy}*}(b, u, q) = 1, \text{ if } \begin{cases} b = b^*_{1,t}(q) \quad \textit{and} \quad \theta_t^{\text{buy}}(b^*_{1,t}(q), b^*_{2,t}(q), q) \le y_t + u, \\ \textit{or} \\ b = b^*_{i,t}(q) \quad \textit{for } i = 2, \dots, I-1 \quad \textit{and} \\ \theta_t^{\text{buy}}(b^*_{i,t}(q), b^*_{i+1,t}(q), q) \le y_t + u < \theta_t^{\text{buy}}(b^*_{i-1,t}(q), b^*_{i,t}(q), q), \\ \textit{or} \\ b = b^*_{I,t}(q) \quad \textit{and} \\ \theta_t^{\text{buy}}(\text{Marginal}_t(q), q) \le y_t + u < \theta_t^{\text{buy}}(b^*_{I,t}(q), b^*_{I-1,t}(q), q), \end{cases} \tag{27}$$

$$d_t^{\text{sell}*}(s, u, q) = 1, \text{ if } \begin{cases} s = s^*_{1,t}(q), \quad \textit{and} \quad y_t + u < \theta_t^{\text{sell}}(s^*_{1,t}(q), s^*_{2,t}(q), q), \\ \textit{or} \\ s = s^*_{j,t}(q), \quad \textit{for } j = 2, \dots, J-1 \quad \textit{and} \\ \theta_t^{\text{sell}}(s^*_{j-1,t}(q), s^*_{j,t}(q), q) \le y_t + u < \theta_t^{\text{sell}}(s^*_{j,t}(q), s^*_{j+1,t}(q), q), \\ \textit{or} \\ s = s^*_{J,t}(q), \quad \textit{and} \\ \theta_t^{\text{sell}}(s^*_{J-1,t}(q), s^*_{J,t}(q), q) \le y_t + u < \theta_t^{\text{sell}}(\text{Marginal}_t(q), q), \end{cases} \tag{28}$$

otherwise,

$$d_t^{\text{buy}*}(b, u, q) = d_t^{\text{sell}*}(s, u, q) = 0. \tag{29}$$

Let $V_t(y_t + u, q)$ be the indirect utility function for a trader at t with valuation $y_t + u$ and quantity q. The indirect utility function is computed by substituting the optimal strategy in equations (27) through (29) into the trader's objective function, equation (12).

Lemma 4. $V_t(y_t + u, q)$ *has the following properties:*

(1) $V_t(y_t + u, q)$ *is a positive, convex function of* $y_t + u$.
(2) *Suppose that* $d_t^{\text{buy}*}(b, u, q) = 1$ *for some* (b, u, q). *Then for* $u' > u$, $V_t(y_t + u', q) > V_t(y_t + u, q)$.
(3) *Suppose that* $d_t^{\text{sell}*}(s, u, q) = 1$ *for some* (s, u, q). *Then for* $u' < u$, $V_t(y_t + u', q) > V_t(y_t + u, q)$.

Figure 2 is an example of $V_t(y_t + u, q)$. Here, $\mathcal{S}_t^*(1) = \{0, 1, 2\}$ and $\mathcal{B}_t^*(1) = \{0, 1, 2\}$: market, one tick, and two tick limit buy and sell orders are optimal for a trader with some valuation and the order quantity is one share. The expected pay-offs as a function of the trader's valuation from submitting sell orders are plotted with dashed lines and the expected pay-offs from submitting buy orders are plotted with dashed–dotted lines. From equation (11), the trader's expected pay-off from submitting any particular order is a linear function of his valuation, with

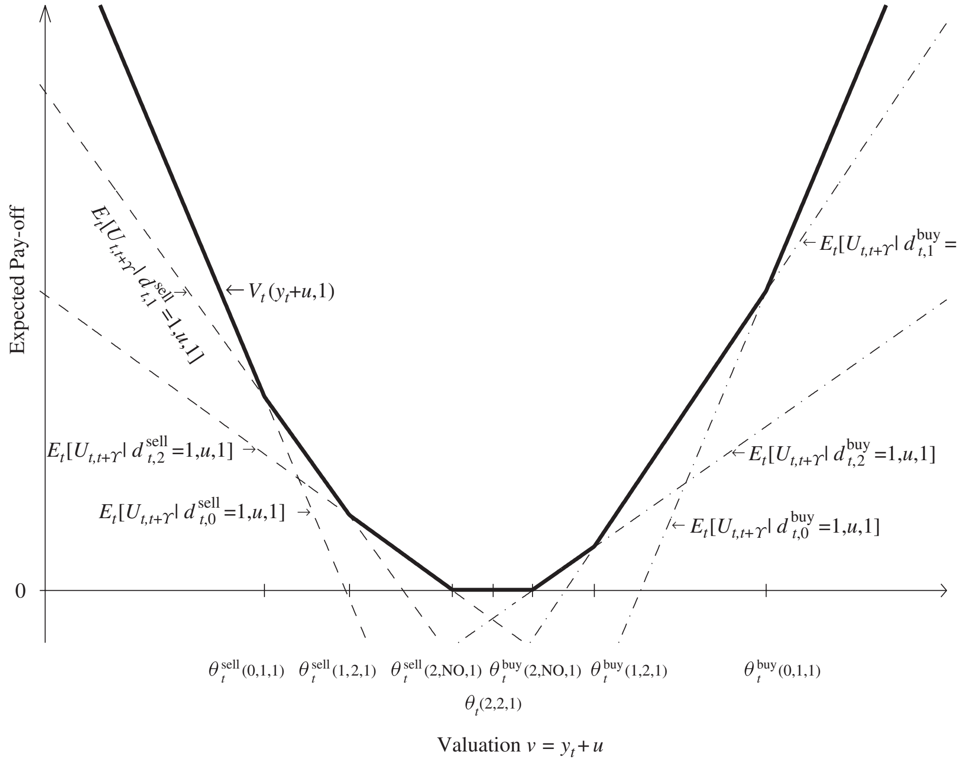

FIGURE 2

Indirect utility function.

The graph is an example of the indirect utility function. The order quantity is set equal to one. The horizontal axis is the trader's valuation, and the vertical axis is the expected pay-off from alternative order submissions. Sell orders are plotted with dashed lines (- - -) and buy orders are plotted with dashed–dotted lines (-.-.). The indirect utility function is plotted with the thick solid line (—). The horizontal axis and the vertical axis have different scale

slope equal to the execution probability for that order. The indirect utility function is plotted with a thick solid line and is the upper envelope of the pay-offs for different order submissions.

A change in the cost of submitting the order, c, leads to a parallel shift in the expected pay-off from all order submissions. A change in the picking off risk for any particular order leads to a parallel shift in the expected pay-off from submitting that particular order, while keeping unchanged the expected pay-off from submitting any other order. A change in the execution probability for any particular order leads to a shift in the slope in the expected pay-off from submitting that particular order, while keeping unchanged the expected pay-off from submitting any other order.

Geometrically, the thresholds are the valuations for which the expected pay-offs intersect. For example, the threshold for a sell market order and a sell limit order at one tick from the best bid quote is $\theta_t^{sell}(0, 1, 1)$; a trader with a valuation less than $\theta_t^{sell}(0, 1, 1)$ submits a sell market order. The thresholds associated with submitting any particular order and submitting no order are the valuations where the expected pay-offs cross the horizontal axis. Here, $\theta_t^{sell}(2, \text{NO}, 1) < \theta_t(2, 2, 1)$ and $\theta_t^{buy}(2, \text{NO}, 1) > \theta_t(2, 2, 1)$. If the trader's valuation is between $\theta_t^{sell}(2, \text{NO}, 1)$ and $\theta_t^{buy}(2, \text{NO}, 1)$, the trader does not submit any order.

We now consider the effects of changing the order submission cost, the picking off risks, and the execution probabilities on the traders' optimal order submission strategy. The order submission cost is a parameter of the model, while the picking risks and the execution probabilities are not exogenous parameters of the model. But the picking off risks and the execution probabilities can change as the traders' information changes.

Consider increasing the order submission cost, c. The expected pay-offs for submitting any order decrease, with all pay-off curves shifting down by the same amount. As a consequence, only the thresholds associated with submitting an order and submitting no order change. The marginal buy threshold increases and the marginal sell threshold decreases; more traders will choose not to make an order submission.

Consider increasing the picking off risk for the one tick sell limit order. The expected pay-offs for submitting a one tick sell limit order decrease, and the expected pay-offs for any other order submissions do not change. The expected pay-offs for a one tick sell limit order make a parallel downward shift, implying that the threshold for the one tick and two tick sell limit orders decreases and the threshold for the one tick sell limit order and sell market order increases. The pay-off curve for a sell market order is steeper than the pay-off curve for the two tick sell limit order; the threshold associated with the market order increases by less than the threshold associated with the two tick order decreases.

Consider increasing the execution probability for the one tick sell limit order. The expected pay-offs for submitting a one tick sell limit order increase, and the expected pay-offs for any other order submissions do not change, implying that the threshold for the one tick and two tick sell limit orders increases and the threshold for the one tick sell limit order and market order decreases.

Suppose that a buy market order arrives and trades against an existing sell limit order. As a consequence, the order book changes, and the traders' information changes. The resulting changes lead to a change in either the best quotes, the execution probabilities, the picking off risks, or some combination of them. Such changes, in turn, lead to a change in the thresholds. The optimal order submission strategy in Figure 2 therefore changes when an order is submitted.

The optimal submission strategy in equations (27)–(29) can be used to compute the probability of a trader submitting a sell order at a price $s^*_{1,t}(q)$ ticks above the current best bid quote, conditional on the arrival of a trader who wishes to trade q shares. Typically, $s^*_{1,t}(q) = 0$—a sell market order is optimal for some trader. The probability is

$$\begin{aligned}\Pr_t(d_t^{\text{sell}*}(s^*_{1,t}(q), u_t, q) = 1 \mid q) &= \Pr_t(y_t + u_t \le \theta_t^{\text{sell}}(s^*_{1,t}(q), s^*_{2,t}(q), q) \mid q) \\ &= G_t(\theta_t^{\text{sell}}(s^*_{1,t}(q), s^*_{2,t}(q), q) - y_t). \qquad (30)\end{aligned}$$

The last line follows from the definition of G_t in equation (4) and because the quantity and the private value are independent random variables. Similarly, for $j = 2, \dots, J-1$

$$\begin{aligned}&\Pr_t(d_t^{\text{sell}*}(s^*_{j,t}(q), u_t, q) = 1 \mid q) \\ &\quad = G_t(\theta_t^{\text{sell}}(s^*_{j,t}(q), s^*_{j+1,t}(q), q) - y_t) - G_t(\theta_t^{\text{sell}}(s^*_{j-1,t}(q), s^*_{j,t}(q), q) - y_t), \quad (31)\end{aligned}$$

and for $j = J$

$$\begin{aligned}&\Pr_t(d_t^{\text{sell}*}(s^*_{J,t}(q), u_t, q) = 1 \mid q) \\ &\quad = G_t(\theta_t^{\text{sell}}(\text{Marginal}_t(q), q) - y_t) - G_t(\theta_t^{\text{sell}}(s^*_{J-1,t}(q), s^*_{J,t}(q), q) - y_t). \quad (32)\end{aligned}$$

Similar expressions to equations (30) through (32) hold for buy orders.

Equations (30) through (32) can be used to interpret previous empirical studies of order submissions. For example, Biais *et al*. (1995) find that a larger ask depth increases the probability of a sell market order and decreases the probability of a sell limit order. A larger ask depth may imply a smaller execution probability for a one tick sell limit order, increasing the threshold valuation for a sell market order and a one tick sell limit order. From equations (30) and (31), increasing the threshold valuation increases the probability of a sell market order and decreases the probability of a one tick sell limit order.

Biais *et al*. (1995) also find that traders are more likely to submit limit orders when the bid–ask spread widens. Suppose the bid–ask spread widens from one to two ticks, holding the best ask quote, the common value, the execution probabilities and picking off risks constant. Such an increase in the spread would decrease the threshold for a sell market order and a one tick sell limit order. From equations (30) and (31), the probability of a sell market order decreases and the probability of a one tick sell limit order increases. A widening of the spread may also increase the execution probability for a one tick sell limit order. An increase in the execution probability would decrease the threshold for a sell market order and a one tick sell limit order. Again, the probability of a sell market order decreases and the probability of a one tick sell limit order increases.

Equations (30) through (32) also provide two ways to interpret the autocorrelation in order submissions documented in existing empirical work. First, the threshold valuations may be autocorrelated if the limit order book changes only gradually causing the expected pay-offs from different order submissions to be autocorrelated. Second, innovations in the common value are correlated with the current order submission, and unless the book adjusts immediately to innovations in the common value, such common value innovations will cause autocorrelated order submissions.

Figure 3 plots the optimal order submission strategy corresponding to the indirect utility function in Figure 2. The distribution of private values G_t is a mixture of three normal distributions. The horizontal axis in the figure is the trader's private value and the vertical axis is the cumulative probability distribution of the private values. The probability of various order submissions is determined by the thresholds, the common value and the distribution of private values G_t by equations (30) through (32).

For example, the probability of a trader submitting a sell market order is $G_t(\theta_t^{\text{sell}}(0,1,1) - y_t)$. A trader with valuation between $\theta_t^{\text{sell}}(1,2,1) - y_t$ and $\theta_t^{\text{sell}}(0,1,1) - y_t$ submits a one tick sell limit order. The probability of a trader submitting a one tick sell limit order is $G_t(\theta_t^{\text{sell}}(1,2,1) - y_t) - G_t(\theta_t^{\text{sell}}(0,1,1) - y_t)$. A trader with a private value between $\theta_t^{\text{sell}}(2,\text{NO},1)$ and $\theta_t^{\text{buy}}(2,\text{NO},1)$ does not submit any order, and the probability of such an event is

$$\Pr_t(\text{NO} \mid q) = G_t(\theta_t^{\text{buy}}(2,\text{NO},1) - y_t) - G_t(\theta_t^{\text{sell}}(2,\text{NO},1) - y_t). \tag{33}$$

Given that a trader may find it optimal to submit no order, the probability of observing a sell market order, conditional on observing any order submission, is

$$\Pr_t(d_t^{\text{sell}*}(s_{1,t}^*(q), u_t, q) = 1 \mid q, \text{order submission}) = \frac{G_t(\theta_t^{\text{sell}}(0,1,1) - y_t)}{1 - \Pr_t(\text{NO} \mid q)}. \tag{34}$$

From equations (30) through (32) the conditional probabilities of observing different order submissions form an ordered qualitative response model, as defined by Amemiya (1985, Definition 9.3.1, p. 292). The conditional choice probabilities can be used to estimate the private value distribution using a sample of order submissions and estimates of the thresholds and the common value. The estimation method for the private values distribution must allow for the possibility that some traders may choose not to make any order submission.

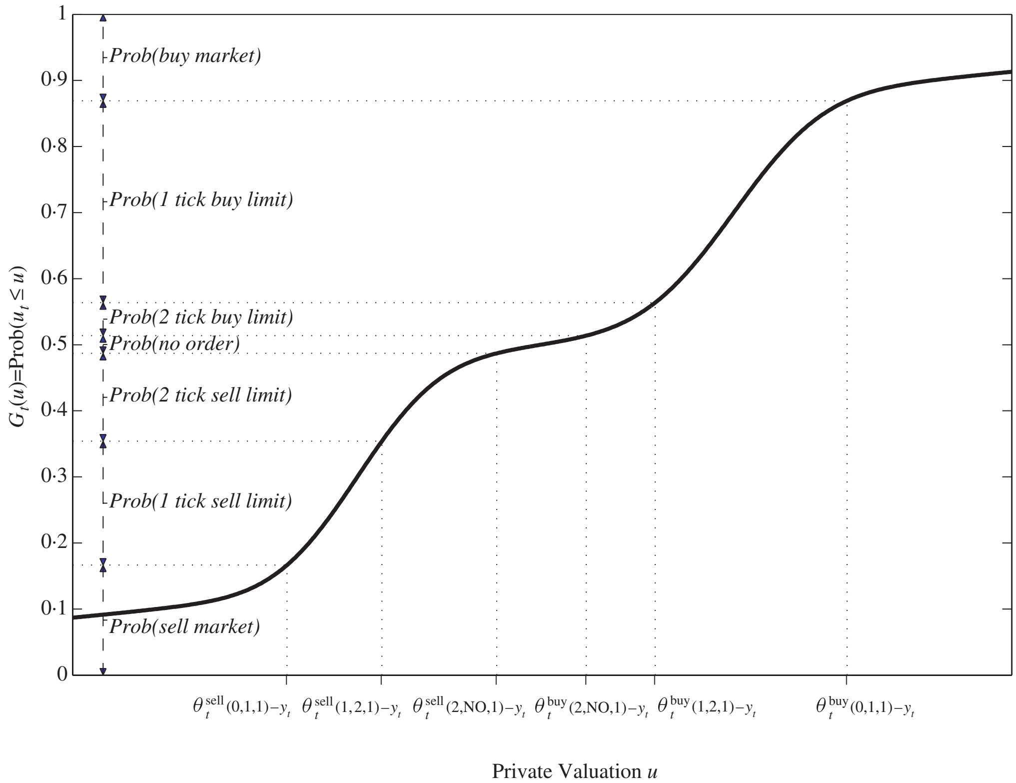

FIGURE 3

Optimal order submission strategy.

The graph is an example of the optimal order submission strategy. The probabilities of observing different order submissions are determined by the threshold valuations, the common value and the distribution of private values. The threshold valuations are computed using equations (16) through (19). The distribution of the private values G_t is a mixture of three normal distributions

3.3. An empirical implication of the optimal order submission strategy

We use Lemma 2 to develop an empirical test of the model. The test does not require knowledge of G_t but does require knowledge of actual order submissions; their prices, execution probabilities and picking off risks. The logic of the test is illustrated by the following example. Suppose the traders submit buy market orders, one tick buy limit orders, and two tick buy limit orders. The best ask quote is 100, and the tick size is 1. The execution probabilities are $\psi_t^{buy}(0,1) = 1$, $\psi_t^{buy}(1,1) = 0{\cdot}7$ and $\psi_t^{buy}(2,1) = 0{\cdot}6$. For simplicity, the picking off risk for all buy limit orders is zero. The order quantity is one share. Observing such data implies that the model is false.

Figure 4 plots the pay-offs for a trader submitting the three different buy orders against the trader's valuation: a dashed line for a buy market order, a light solid line for a one tick buy limit order, and a dashed–dotted line for a two tick buy limit order. There is no valuation for which the one tick buy limit order is optimal; the expected pay-off from submitting a one tick buy limit order is always lower than the pay-offs from submitting a buy market order or a two tick buy

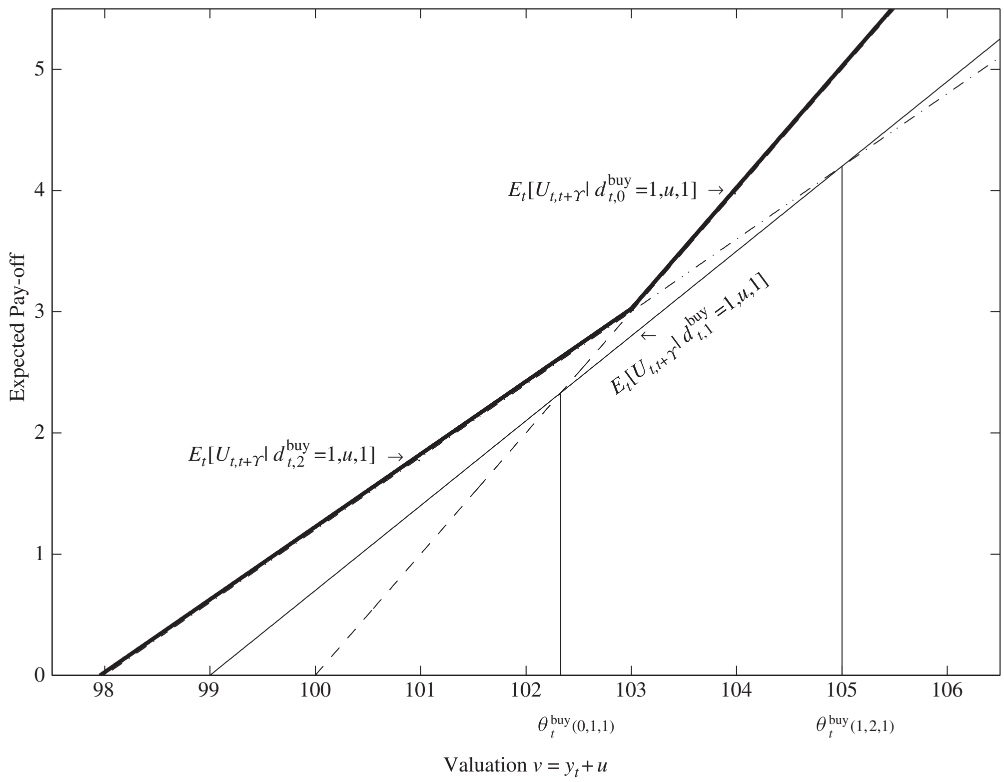

FIGURE 4

Example of violation of the monotonicity of the thresholds.

The figure is an example where the threshold valuations do not satisfy the monotonicity condition $\theta^{buy}_t(0, 1, 1) < \theta^{buy}_t(1, 2, 1)$. The execution probabilities for limit orders are monotonically decreasing in the distance between the limit order price and the best ask quote. The execution probabilities are $\psi^{buy}_t(0, 1) = 1$, $\psi^{buy}_t(1, 1) = 0{\cdot}7$, $\psi^{buy}_t(2, 1) = 0{\cdot}6$; the tick size is 1; the best ask quote is 100; and the picking off risks are equal to zero. The expected pay-off for a trader submitting a buy market order (- - -), a one tick buy limit order (—) and a two tick buy limit order (-.-.) are plotted as a function of the trader's valuation

limit order. The threshold valuations are

$$\begin{aligned} \theta^{buy}_t(0, 1, 1) &= 100 + \frac{(1)(0{\cdot}7)}{(1{\cdot}0 - 0{\cdot}7)} = 102{\cdot}33, \\ \theta^{buy}_t(1, 2, 1) &= 99 + \frac{(1)(0{\cdot}6)}{(0{\cdot}7 - 0{\cdot}6)} = 105{\cdot}00. \end{aligned} \tag{35}$$

In the example, $\theta^{buy}_t(1, 2, 1) > \theta^{buy}_t(0, 1, 1)$—the threshold valuations violate the monotonicity restriction. Since some traders submit a one tick buy limit order, the observed order submissions are not the outcome of the optimization problem in equations (12) through (14). Computing the threshold valuations requires only the execution probabilities for the orders that are actually submitted. It does not require knowledge of the traders' valuations or of the distribution of the traders' valuations, or knowledge of the execution probabilities and picking off risks of orders not submitted by the traders.

The example is not a knife-edge case. Hold the execution probability for a buy market order equal to one and the execution probability for a two tick buy limit order equal to 0·6. With execution probabilities for a one tick buy limit order between 0·6 and 0·75, the model is inconsistent with traders submitting a one tick buy limit order. With execution probabilities for a one tick limit order between 0·75 and 1·00, the model is consistent with traders submitting a one tick buy limit order.

4. EMPIRICAL EVIDENCE

Lemma 2 states that if the traders solve the optimization problem in equations (12)–(14), then the thresholds evaluated at the orders chosen by the traders form a monotone sequence when the orders are ranked according to the execution probabilities. We identify orders chosen by the traders with positive probability in our sample and rank them according to their execution probabilities. We compute the thresholds at the orders and test the monotonicity property.

We assume that the traders' conditioning information is measured by a vector of conditioning variables. Since the model imposes weak functional form restrictions on the execution probabilities and picking off risks, we estimate the traders' expectations using non-parametric regressions of the realized fill history of each of the orders onto a set of conditioning variables. Ideally, we would include in the conditioning information the order quantity; the entire limit order book; and information that predicts the distribution of the common value innovations, the distribution of private values and the cancellation time. It is neither feasible nor computationally tractable to do so. Instead, we condition on six variables: order quantity, ask depth, bid depth, lagged volume, index volatility and the time of day. Table 4 contains definitions of the conditioning variables. Let X_t denote the conditioning information. In unreported ordered probit models, we reject the null that the conditioning variables do not predict the traders' order submissions.[1]

We include the order quantity since the trade-offs between order prices, execution probabilities and picking off risks are likely to depend on order quantity. Harris and Hasbrouck (1996) show that on the New York Stock Exchange, an increase order quantity tends to decrease execution probabilities.

We include the ask and bid depth to capture competition between traders on the same side of the limit order book. The book quantities reported in Table 3 indicate that the depth is volatile in our sample. We do not condition on the bid–ask spread, since it is relatively constant over the sample. The time priority rule implies that a new sell limit order at the best ask quote is likely to have a lower execution probability when the ask depth is large. The depth on the other side of the book may also influence the order submissions. A larger bid depth implies more competition among buyers, increasing the probability of buy market order submissions in the future and therefore increasing the execution probabilities for new sell limit orders.

We include lagged trading volume to measure recent order submission activity. If the arrival of traders to the market is clustered in time, lagged trading volume, or other measures of lagged activity, are useful in predicting execution probabilities and picking off risk. Biais *et al*. (1995) find empirically that trading activity is clustered in time. As a consequence, lagged trading volume may have an impact on the execution probabilities and picking off risks.

We include lagged index volatility to measure volatility. In Foucault's (1999) theoretical model, the volatility of the asset influences both the execution probabilities and the picking off risks. Such effects are likely to carry over to many formulations of the order submission problem. We condition on lagged index volatility to measure volatility for two reasons. First,

1. For buy orders: $\chi^2_6 = 2428{\cdot}2$, with p-value $= 0{\cdot}00$ and for sell orders: $\chi^2_6 = 1622{\cdot}7$, with p-value $= 0{\cdot}00$.

TABLE 4

Description of the conditioning variables

Conditioning variable	Description
Order quantity	The number of shares in the order submitted at t. Transformed by taking the logarithm of the number of shares.
Ask depth	The number of shares in the order book at the best ask quote. Transformed by taking the logarithm of one plus the number of shares divided by 1000.
Bid depth	The number of shares in the order book at the best bid quote. Transformed by taking the logarithm of one plus the number of shares divided by 1000.
Lagged volume	The cumulative number of shares transacted during the minimum of the previous 10 minutes and the time since the market opening. Transformed by taking the logarithm of one plus the number of shares divided by 1000.
Index volatility	The standard deviation of the OMX index returns over the minimum of the 60 minutes and the number of minutes since market opening. Transformed by taking the logarithm of one plus the volatility times $\sqrt{270}$; there are 270 trading minutes per day.
Time of day	The time since the market opening. Measured in seconds since midnight. Transformed by dividing by 3600.

The table describes the construction of the conditioning variables. All variables are transformed to have approximately the same scale.

index volatility measures market-wide volatility and may be less sensitive to volatility induced by transactions randomly occurring at the best bid or ask quotes than volatility measured by Ericsson's transaction price volatility. Second, conditional volatility is autocorrelated.

We include the time of day because market closing may cause a deadline effect. For example, traders may not wish to have limit orders outstanding overnight because overnight news may cause discrete price movements and therefore execution probabilities and picking off risks may vary with the time until the next market close. Biais *et al.* (1995) find empirically that order submissions depend on the time until the next market closing.

4.1. *Test of strictly positive conditional choice probabilities*

Define $\ddot{B}(X_t)$ as the set of indexes for buy order prices that are chosen with strictly positive probability in our sample conditional on X_t, with elements ordered by the distance from the best ask quote $\ddot{b}_1 < \ddot{b}_2 < \cdots < \ddot{b}_N$:

$$\ddot{B}(X_t) = \left\{b \mid \Pr(d_{t,b}^{\text{buy}} = 1 \mid X_t) > 0\right\}. \tag{36}$$

Define $\ddot{S}(X_t)$ similarly.

An order is in $\ddot{B}(X_t)$ or $\ddot{S}(X_t)$ for all X_t, if it has strictly positive conditional choice probability for all X_t. Suppose that buy orders at $n, n+1, \ldots, N$ ticks from the best ask quote and sell orders at $o, o+1, \ldots, O$ ticks from the best bid quote all have conditional choice probabilities greater than or equal to LB, where $LB > 0$. Let z_t^{++} be a vector of strictly positive measurable functions of the vector X_t, and $\otimes$ the Kronecker product. Define

$$PC = E\left[\begin{pmatrix} d_{t,n}^{\text{buy}} - LB \\ \vdots \\ d_{t,N}^{\text{buy}} - LB \\ d_{t,o}^{\text{sell}} - LB \\ \vdots \\ d_{t,O}^{\text{sell}} - LB \end{pmatrix} \otimes z_t^{++}\right]. \tag{37}$$

The law of iterated expectations and the restriction that the conditional choice probabilities are greater than or equal to LB imply the null hypothesis

$$H_0 : PC \geqslant 0. \tag{38}$$

We use the sample moment analogue $\widehat{PC}_T$ to form an estimator for PC. Under standard conditions, $\sqrt{T}(\widehat{PC}_T - PC)$ converges in distribution to a normal random variable, with asymptotic variance–covariance matrix, $\mathcal{A}_{PC}$. Wolak (1989) derives a test statistic for a local test of H_0,

$$M_{PC} = \min_{\{a|a\geq 0\}} T(\widehat{PC}_T - a)\mathcal{A}_{PC}^{-1}(\widehat{PC}_T - a)', \tag{39}$$

and shows that under H_0, M_{PC} converges in distribution to the weighted sum of χ^2 variables,

$$\Pr(M_{PC} \geq r) = \sum\nolimits_{k=0}^{\dim(\mathcal{A}_{PC})} \Pr[\chi_k^2 \geq r]\text{w}(\dim(\mathcal{A}_{PC}), \dim(\mathcal{A}_{PC}) - k, \mathcal{A}_{PC}), \tag{40}$$

where χ_k^2 is a χ^2 variable with k degrees of freedom, $\dim(\mathcal{A}_{PC})$ is the rank of the asymptotic variance–covariance matrix, and $\text{w}(\dim(\mathcal{A}_{PC}), \dim(\mathcal{A}_{PC}) - k, \mathcal{A}_{PC})$, for $k = 0, \ldots, \dim(\mathcal{A}_{PC})$ are weights that depend on the asymptotic variance–covariance matrix. Wolak (1989) describes a Monte Carlo method for calculating the weights.

Table 5 reports the results for the tests that the conditional choice probabilities are greater than or equal to 0·02. The tests are computed for one tick, two tick and three tick buy and sell limit orders. Table 2 reports that in our sample, approximately 48% of the orders submitted are market orders, and so we do not include market orders in the test.

Each row reports the point estimates of the unconditional differences in decision indicators and 0·02 multiplied by positive instruments, the associated standard errors and p-values for the null of positive conditional choice probabilities for different order submissions.[2] Each column corresponds to a different positive instrument. The final row of the table reports the M_{PC} test described above for each instrument and all submissions, and the final column of the table reports the test statistic across the instruments. All of the point estimates are strictly positive and none of the tests reject the null hypothesis of strictly positive conditional choice probabilities; the p-values are all greater than 0·98.

We find no evidence against the hypothesis that the traders submit one tick, two tick and three tick limit orders at each value of the conditioning variables. We compute the thresholds for market orders and limit orders up to three ticks away from the quotes. A censoring bias could arise in our subsequent tests if some order submission between a market and three tick limit order were optimal for some trader, but not used in computing the thresholds. Since we use all order

2. The standard errors are computed with 50 lags using the Newey and West (1987) procedure. The empirical results are robust to changes in the lag length. The asymptotic p-values for the monotonicity tests are computed using 10,000 Monte Carlo simulation trials.

TABLE 5

Test of strictly positive conditional choice probabilities

Order	Instruments							Joint M_{PC} statistic
	Constant	Order quantity	Ask depth	Bid depth	Lagged volume	Index volatility	Time of day	
				Buy limit orders				
1 Tick	0·18	1·27	0·35	0·37	0·45	0·12	2·20	0·00
	(0·00)	(0·03)	(0·01)	(0·01)	(0·01)	(0·00)	(0·06)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
2 Ticks	0·03	0·20	0·04	0·04	0·07	0·02	0·32	0·00
	(0·00)	(0·01)	(0·00)	(0·00)	(0·01)	(0·00)	(0·02)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
3 Ticks	0·02	0·13	0·04	0·04	0·05	0·02	0·26	0·00
	(0·00)	(0·01)	(0·00)	(0·00)	(0·01)	(0·00)	(0·03)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Sell limit orders				
1 Tick	0·13	1·00	0·32	0·37	0·33	0·09	1·61	0·00
	(0·00)	(0·03)	(0·01)	(0·01)	(0·01)	(0·00)	(0·05)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
2 Ticks	0·02	0·15	0·03	0·04	0·05	0·01	0·21	0·00
	(0·00)	(0·01)	(0·00)	(0·00)	(0·01)	(0·00)	(0·02)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
3 Ticks	0·01	0·04	0·02	0·02	0·02	0·01	0·08	0·00
	(0·00)	(0·01)	(0·00)	(0·00)	(0·00)	(0·00)	(0·02)	(0·00)
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Joint M_{PC} statistic				
All limit orders	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·98	0·99	0·98	0·98	0·98	0·99	0·98	1·00

The top two panels of the table report the averages of the differences of the order choices and 0·02, multiplied by positive instruments. Asymptotic standard errors in parentheses and p-values are reported below each point estimate. The rightmost column and the bottom panel of the table report joint M_{PC} test statistics across the instruments, the order prices, and across instruments and order prices, with p-values reported below each test statistic. We ensure that all instruments are strictly positive by replacing them with 0·00001 if they are zero.

submissions between a market order and a three tick limit order, our tests do not face such a censoring bias.

4.2. *Test of monotonicity of the execution probabilities*

If execution probabilities are monotone in the distance from the best quotes, then ranking the order prices by the distance from the best quotes is equivalent to ranking them by their execution probabilities. The assumption is a weak one in a deep market—a deep market imposes enough competitive pressure on traders such that they cannot increase execution probabilities by placing less aggressive orders.

The execution probabilities are computed as a non-parametric regression of realized fills on information known at the time of order submission. Let $\mathcal{K}$ be a multidimensional kernel function and h_T a bandwidth associated with each argument. The non-parametric estimate of $\psi^{\text{sell}}(\ddot{s}, X_t)$ is

$$\widehat{\psi}^{\text{sell}}(\ddot{s}, X_t) \equiv \frac{\sum_{t' \neq t}^{T} \left(d_{t',\ddot{s}}^{\text{sell}} \sum_{\tau=0}^{\Upsilon} \frac{dQ_{t',t'+\tau}}{q_{t'}} \right) \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}{\sum_{t' \neq t}^{T} \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}, \tag{41}$$

for $\ddot{s} \in \ddot{S}(X_t)$, with a similar definition on the buy side. From the definition of $\ddot{S}(X_t)$, $\widehat{\psi}^{\text{sell}}(\ddot{s}, X_t)$ is well defined. Since the lifetime of most limit orders is less than 2 days in our sample, we set the maximum lifetime of the order, Υ in equation (41), equal to 2 days.

To test monotonicity of the execution probabilities, define

$$DF \equiv E\left[I(X_t \in \bar{X}) \begin{pmatrix} \psi^{\text{buy}}(\ddot{b}_1, X_t) - \psi^{\text{buy}}(\ddot{b}_2, X_t) \\ \psi^{\text{buy}}(\ddot{b}_2, X_t) - \psi^{\text{buy}}(\ddot{b}_3, X_t) \\ \vdots \\ \psi^{\text{sell}}(\ddot{s}_1, X_t) - \psi^{\text{buy}}(\ddot{s}_2, X_t) \\ \psi^{\text{sell}}(\ddot{s}_2, X_t) - \psi^{\text{sell}}(\ddot{s}_3, X_t) \\ \vdots \end{pmatrix} \otimes z_t^{++} \right], \tag{42}$$

where $I(X_t \in \bar{X})$ is a trimming indicator for the set $\bar{X}$ in the interior of the support of X_t. The trimming indicator is used to simplify the asymptotic distribution. Applying the law of iterated expectations, monotonicity of the execution probabilities implies the null hypothesis

$$H_1 : DF > 0. \tag{43}$$

We use the sample moment analogue of DF to form the estimator $\widehat{DF}_T$, using the non-parametric estimators of the execution probabilities. In Appendix C, we provide regularity conditions under which $\sqrt{T}(\widehat{DF}_T - DF)$ converges in distribution to a normal random variable, and we provide the asymptotic variance–covariance matrix, $\mathcal{A}_{DF}$. We form a similar test statistic to M_{PC} in equation (39) above as a test of H_1.

Table 6 reports the results of the monotonicity tests of the execution probabilities. The tests are computed using the execution probabilities for market and one tick limit orders; one and two tick limit orders; and two and three tick limit orders, for both buy and sell orders.

Each row reports the point estimates of the unconditional differences in execution probabilities multiplied by positive instruments, standard errors and p-values for the null of monotonicity of the execution probabilities for different order submissions.[3] Each column corresponds to a different positive instrument. The final row of the table reports the M_{DF} test described above for each instrument and all submissions, and the final column of the table reports the test statistic across each order submission. All of the point estimates are strictly positive and none of the tests reject the null hypothesis of monotonicity of the execution probabilities; the p-values are all greater than 0·98.

Together, the test statistics reported in Tables 5 and 6 fail to reject that buy and sell market orders, one tick, two tick and three tick limit orders are chosen for each value of the conditioning information, and that ordering the orders by the distance from the quotes is the same as ordering them by their execution probabilities. We use the associated threshold valuations to form a monotonicity test.

4.3. Computing the threshold valuations

To compute the threshold valuations, we need estimates of the picking off risks. To form estimates of the picking off risks, we need estimates of changes in the common value. A common proxy for the common value in many microstructure applications is the mid-quote, equal to the average of the best bid and ask quote. Such a proxy is inappropriate in our application for two reasons. First, a trader influences the current mid-quote by his order submission at time t. Second,

3. The standard errors are computed with 50 lags using the method described in Appendix C to capture the overlap in the errors in the execution probabilities between orders submitted at different times. The empirical results are robust to changes in the lag length. The asymptotic p-values for the monotonicity tests are computed using 10,000 Monte Carlo simulation trials.

TABLE 6

Monotonicity tests for the execution probabilities

Order	Constant	Order quantity	Ask depth	Bid depth	Lagged volume	Index volatility	Time of day	Joint M_{PC} statistic
	Instruments							
				Buy limit orders				
Market	0·31	2·13	0·68	0·76	0·75	0·20	3·70	0·00
−1 Tick limit	(0·02)	(0·11)	(0·04)	(0·04)	(0·04)	(0·01)	(0·19)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
1 Tick limit	0·33	2·29	0·72	0·79	0·82	0·21	3·99	0·00
−2 Tick limit	(0·02)	(0·16)	(0·06)	(0·05)	(0·06)	(0·02)	(0·28)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
2 Tick limit	0·20	1·35	0·42	0·47	0·49	0·13	2·36	0·00
−3 Tick limit	(0·02)	(0·16)	(0·05)	(0·05)	(0·06)	(0·02)	(0·29)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Sell limit orders				
Market	0·35	2·44	0·77	0·85	0·86	0·22	4·23	0·00
−1 Tick limit	(0·02)	(0·13)	(0·05)	(0·05)	(0·05)	(0·02)	(0·20)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
1 Tick limit	0·35	2·37	0·77	0·85	0·85	0·22	4·17	0·00
−2 Tick limit	(0·02)	(0·18)	(0·06)	(0·06)	(0·07)	(0·02)	(0·28)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
2 Tick limit	0·13	0·92	0·28	0·32	0·33	0·09	1·60	0·00
−3 Tick limit	(0·03)	(0·19)	(0·06)	(0·06)	(0·08)	(0·02)	(0·29)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Joint M_{PC} statistic				
All limit orders	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·98	0·99	0·99	0·98	0·98	0·98	0·99	1·00

The top two panels of the table report the average differences of the execution probability for different order prices multiplied by positive instruments. The asymptotic standard errors in parentheses and *p*-values are reported below the point estimates. The rightmost column and the bottom panel of the table report joint M_{DF} test statistics across the instruments, the order prices, and across instruments and order prices, with *p*-values reported below each test statistic. We ensure that all instruments are strictly positive by replacing them with 0·00001 if they are zero.

a limit order is executed once it becomes the best bid or ask quote, inducing a mechanical correlation between the mid-quote and the execution. Our common value is not based on the mid-quote but instead is based on the level of the market index.

From equation (3), the common value is integrated of order one, or $I(1)$. We assume that there is an $I(1)$ vector of factors, f_t, such that

$$y_t = \beta f_t, \tag{44}$$

with β a parameter.

The best bid quote is observed when there are buy limit orders outstanding in the order book. Accordingly, denote by t' a time period where there are outstanding buy limit orders in the book. We provide conditions in Appendix B for the best bid quote to be cointegrated with the common value,

$$\begin{aligned} p^{\text{sell}}_{t',0} &= y_{t'} + \varepsilon_{t'} \\ &= \beta f_{t'} + \varepsilon_{t'}, \end{aligned} \tag{45}$$

where $p^{\text{sell}}_{t',0}$ is the best bid quote at time t' and $\varepsilon_{t'}$ is $I(0)$. Let $\widehat{\beta}_{T'}$ denote the least squares estimate of β obtained by regressing $p^{\text{sell}}_{t',0}$ on $f_{t'}$. We form an estimate of changes in the the common value as

$$\widehat{y}_{t+\tau} - \widehat{y}_t = \widehat{\beta}_{T'}(f_{t+\tau} - f_t). \tag{46}$$

TABLE 7

Common value estimation

	Unit root test	Cointegrating regression: Coefficient	Cointegrating regression: Cointegration test
OMX index	$-0{\cdot}89$		
	p-value $= 0{\cdot}79$		
Bid quote	$-1{\cdot}11$	$0{\cdot}37$	$-3{\cdot}67$
	p-value $= 0{\cdot}71$	$(0{\cdot}00)$	p-value $< 0{\cdot}03$
Ask quote	$-1{\cdot}08$	$0{\cdot}36$	$-3{\cdot}62$
	p-value $= 0{\cdot}72$	$(0{\cdot}00)$	p-value $< 0{\cdot}03$

The first column reports unit root tests for the best bid quote, best ask quote and the OMX index. All series are demeaned. There are 20,760 observations. The unit root test is an augmented Dickey–Fuller t-test with 10 lags, and p-values are reported below each t-statistic. The second column reports the estimated coefficient on the demeaned OMX index using both quote series with the standard error in parentheses. The final column reports an augmented Engle–Granger test for cointegration computed using 10 lags.

We used minute-by-minute observations of the value of the OMX index as our factor series. The OMX index is a value-weighted index of the 30 most traded companies on the Stockholm Stock Exchange. Bossaerts (1988) provides conditions for such cointegration to hold in standard asset pricing models such as the Capital Asset Pricing Model. We also experimented with including the daily sampled \$/SKr exchange rate and daily sampled Swedish interest rates as factors. The exchange rate and interest rates added little explanatory power and we therefore only report the results obtained using the market index.

The first column of Table 7 reports a Dickey–Fuller test statistic for the null hypothesis of a unit root in the OMX index, the bid quote and the ask quote; the test fails to reject the null. The final two columns of Table 7 report our estimate of $\widehat{\beta}_{T'}$ and an Engle–Granger cointegration test. We reject the null hypotheses that the bid and the ask are not cointegrated with the OMX index.

Our estimator for $\xi^{\text{sell}}(\ddot{s}, X_t)$ is

$$\widehat{\xi}^{\text{sell}}(\ddot{s}, X_t) = \frac{\sum_{t' \neq t}^{T} \left(d_{t',\ddot{s}}^{\text{sell}} \sum_{\tau=0}^{\Upsilon} \frac{dQ_{t',t'+\tau}}{q_{t'}} (\widehat{y}_{t'+\tau} - \widehat{y}_{t'}) \right) \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}{\sum_{t' \neq t}^{T} \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}, \tag{47}$$

where $\widehat{y}_{t+\tau} - \widehat{y}_t$ is the estimate of changes in the common value in (46), and $\ddot{s} \in \ddot{S}(X_t)$. We form a similar estimator for buy order picking off risks.

We estimate the threshold valuations by

$$\widehat{\theta}^{\text{sell}}(\ddot{s}, \ddot{s}', X_t) = p_{t,\ddot{s}}^{\text{sell}} - \frac{(p_{t,\ddot{s}'}^{\text{sell}} - p_{t,\ddot{s}}^{\text{sell}})\widehat{\psi}^{\text{sell}}(\ddot{s}', X_t) + (\widehat{\xi}^{\text{sell}}(\ddot{s}, X_t) - \widehat{\xi}^{\text{sell}}(\ddot{s}', X_t))}{\widehat{\psi}^{\text{sell}}(\ddot{s}, X_t) - \widehat{\psi}^{\text{sell}}(\ddot{s}', X_t)}, \tag{48}$$

with a similar estimator for the buy side. If $\psi^{\text{sell}}(\ddot{s}, X_t) - \psi^{\text{sell}}(\ddot{s}', X_t) > 0$, then $\theta^{\text{sell}}(\ddot{s}, \ddot{s}', X_t)$ is a continuous function of the execution probabilities and picking off risks. Consistency of the estimators for the execution probabilities and picking off risks therefore implies consistency of the estimator $\widehat{\theta}^{\text{sell}}(\ddot{s}, \ddot{s}', X_t)$ for the thresholds.

4.4. Test of monotonicity of the threshold valuations

We use our estimators for the threshold valuations, equation (48), to form a test statistic for the monotonicity restrictions in equation (23) of Lemma 2. If

$$\{0, 1, 2, 3\} \subset \mathcal{S}^*(X_t), \qquad \text{and} \qquad \{0, 1, 2, 3\} \subset \mathcal{B}^*(X_t), \tag{49}$$

Lemma 2 implies

$$\theta^{\text{buy}}(0, 1, X_t) > \theta^{\text{buy}}(1, 2, X_t) > \theta^{\text{buy}}(2, 3, X_t), \tag{50}$$

$$\theta^{\text{sell}}(2, 3, X_t) > \theta^{\text{sell}}(1, 2, X_t) > \theta^{\text{sell}}(0, 1, X_t), \tag{51}$$

and

$$\theta^{\text{buy}}(2, 3, X_t) > \theta^{\text{sell}}(2, 3, X_t). \tag{52}$$

Define

$$D\theta \equiv E\left[I(X_t \in \bar{X}) \begin{pmatrix} \theta^{\text{buy}}(0, 1, X_t) - \theta^{\text{buy}}(1, 2, X_t) \\ \theta^{\text{buy}}(1, 2, X_t) - \theta^{\text{buy}}(2, 3, X_t) \\ \theta^{\text{sell}}(1, 2, X_t) - \theta^{\text{sell}}(0, 1, X_t) \\ \theta^{\text{sell}}(2, 3, X_t) - \theta^{\text{sell}}(1, 2, X_t) \\ \theta^{\text{buy}}(2, 3, X_t) - \theta^{\text{sell}}(2, 3, X_t) \end{pmatrix} \otimes z_t^{++} \right]. \tag{53}$$

Inequalities (50) through (52) and the law of iterated expectations imply the null hypothesis,

$$H_2: \; D\theta > 0. \tag{54}$$

We use the sample moment analogue of $D\theta$ to form the estimator $\widehat{D\theta}_T$, using the non-parametric estimators for the threshold valuations. In Appendix C, we provide conditions under which $\sqrt{T}(\widehat{D\theta}_T - D\theta)$ converges in distribution to a normal random variable and provide the asymptotic variance–covariance matrix. We form a similar test statistic to M_{PC} in equation (39) above as a test of H_2.

Table 8 reports estimates of the average threshold valuation differences. The top panel reports the average of the differences for buy orders multiplied by positive instruments; reported below each estimate are associated asymptotic standard errors and p-values for the null that the differences are positive.[4] Each column uses a different positive instrument. The final column reports the $M_{D\theta}$ statistic for each difference for all the instruments jointly, with asymptotic p-values reported in parentheses. The second panel reports estimates of the differences for sell orders. The point estimates of the threshold valuation differences are positive for all buy and sell order thresholds and the tests do not reject the null hypothesis of monotonicity, either individually for each pair of threshold valuations and instrument, or jointly across all instruments.

The third panel reports estimates of the differences between the threshold valuation for a two tick and a three tick buy limit order and the threshold valuation for a two and a three tick sell limit order. The point estimates are negative for all instruments. The associated tests all reject the null hypothesis of monotonicity at the 5% level. The joint test across all instruments reported in the last column rejects the null hypothesis at the 1% level.

The bottom panel of Table 8 reports the joint tests for the buy threshold valuation differences, the sell threshold valuation differences and the buy and sell threshold valuations together with asymptotic p-values reported below the point estimates. For all instruments, we fail to reject the null hypothesis of monotonicity for the buy and the sell thresholds separately.

4. The standard errors are computed as described in Appendix C using the Newey and West (1987) procedure with 50 lags, and the asymptotic p-value for the $M_{D\theta}$ statistic is computed using the simulation method given in Wolak (1989) with 10,000 Monte Carlo simulation trials. The results are robust to changes in the lag length.

TABLE 8

Monotonicity tests for the threshold valuations

Threshold valuation difference	Constant	Order quantity	Ask depth	Bid depth	Lagged volume	Index volatility	Time of day	Joint M_{PC} statistic
				Instruments				
				Buy threshold valuations				
$\theta^{\text{buy}}(0, 1, X_t)$	2·15	14·82	4·66	5·11	5·38	1·42	25·76	0·00
$-\theta^{\text{buy}}(1, 2, X_t)$	(0·15)	(1·03)	(0·37)	(0·40)	(0·40)	(0·12)	(1·84)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
$\theta^{\text{buy}}(1, 2, X_t)$	1·21	8·34	2·73	3·02	2·93	0·78	14·58	0·00
$-\theta^{\text{buy}}(2, 3, X_t)$	(0·14)	(0·93)	(0·33)	(0·35)	(0·38)	(0·12)	(1·68)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Sell threshold valuations				
$\theta^{\text{sell}}(1, 2, X_t)$	2·02	13·89	4·58	5·06	5·02	1·32	24·28	0·00
$-\theta^{\text{sell}}(0, 1, X_t)$	(0·16)	(1·20)	(0·38)	(0·44)	(0·44)	(0·14)	(1·83)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
$\theta^{\text{sell}}(2, 3, X_t)$	0·24	1·61	0·47	0·54	0·57	0·16	2·82	0·00
$-\theta^{\text{sell}}(1, 2, X_t)$	(0·49)	(3·61)	(1·14)	(1·29)	(1·44)	(0·41)	(5·48)	
	0·69	0·67	0·66	0·66	0·65	0·65	0·70	0·99
				Buy and sell threshold valuations				
$\theta^{\text{buy}}(2, 3, X_t)$	−1·43	−9·84	−3·26	−3·62	−3·47	−0·92	−17·27	13·16
$-\theta^{\text{sell}}(2, 3, X_t)$	(0·42)	(3·01)	(1·00)	(1·11)	(1·25)	(0·38)	(4·77)	
	0·00	0·00	0·00	0·00	0·00	0·01	0·00	0·01
				Joint $M_{D\theta}$ statistic				
Buy thresholds	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·75	0·76	0·74	0·75	0·75	0·76	0·75	1·00
Sell thresholds	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·75	0·75	0·75	0·75	0·75	0·75	0·75	1·00
Buy and sell	70·35	76·33	80·97	76·43	50·46	32·90	79·49	99·31
thresholds	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00

The top three panels of the table report the average differences of threshold valuations for different order prices multiplied by positive instruments. Asymptotic standard errors in parentheses and the p-values are reported below the point estimates. The rightmost column and the bottom panel of the table report joint $M_{D\theta}$ test statistics across the instruments, the order prices, and across instruments and order prices, with p-values reported below each test statistic. We ensure that all instruments are strictly positive by replacing them with 0·00001 if they are zero.

The final two rows of the table test the monotonicity of all thresholds jointly, and the tests all reject the null hypothesis at the 1% level.

Figure 5 plots the estimated pay-offs for buy and sell market, one, two and three tick limit orders, evaluated at the observation in the sample where the conditioning variables are closest to their sample averages. At this observation, the best bid quote is 116 SKr and the best ask quote is 117 SKr: $p_{t,0}^{\text{sell}} = 116$ SKr and $p_{t,0}^{\text{buy}} = 117$ SKr. The estimated pay-offs for traders with valuations equal to the threshold valuations are computed by substituting estimates of the threshold valuations, the execution probabilities and picking off risks, and the order quantity into equation (11), and dividing by the order quantity. The order submission cost per share, c, is set to zero.

The estimated pay-offs for traders with valuations between the threshold valuations lie on the linear segment between the estimated pay-offs at the threshold valuations. The top plot is the expected pay-offs for buy orders and the bottom plot is the expected pay-offs for sell orders. The horizontal axis is the trader's valuation and the vertical axis is the expected pay-off. The thick solid line is the maximum obtainable pay-offs, if the traders were constrained to submit buy orders or sell orders only.

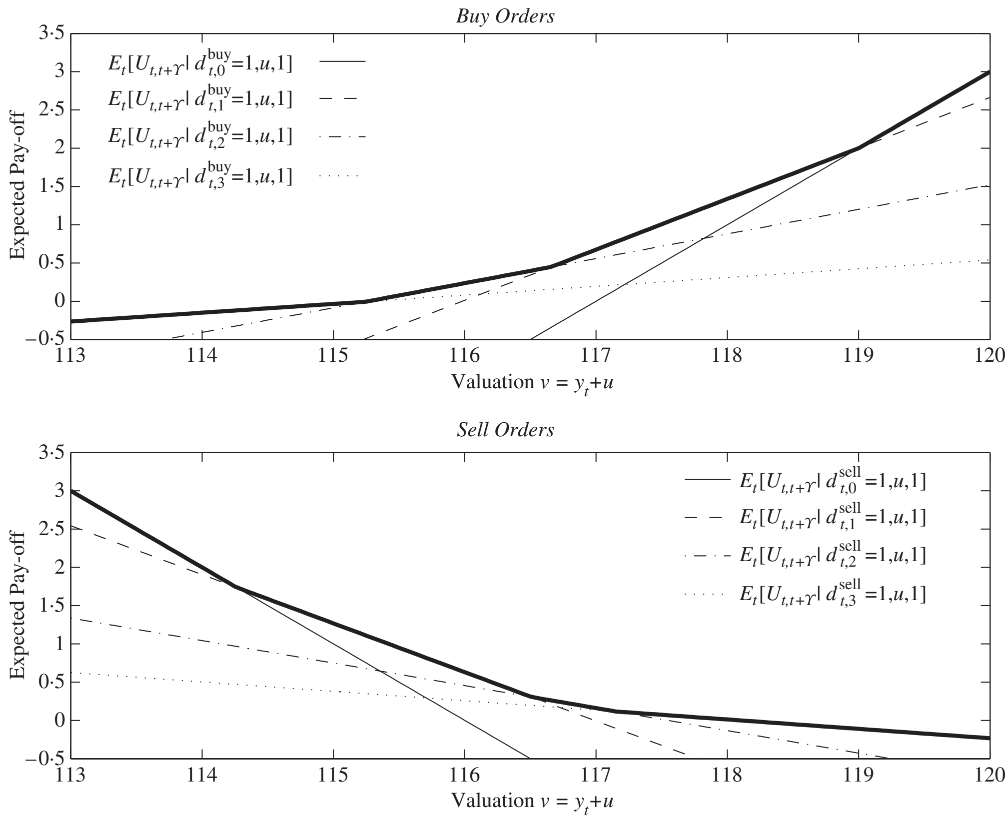

FIGURE 5

Estimated pay-offs.

The figure plots the estimated pay-offs as a function of the trader's valuation. The estimated pay-offs are evaluated at the sample observation with conditioning variables closest to their sample averages. The horizontal axis gives the trader's valuation and the vertical axis the pay-offs for alternative order submissions

The threshold valuations satisfy the monotonicity restriction for buy order submissions and sell order submissions separately. The threshold valuations do not satisfy the monotonicity restrictions for buy order submissions and sell order submissions jointly. Suppose that traders were restricted to submit buy orders only. The optimal buy order for a trader with a valuation of 115 would be to submit a three tick buy limit order. But if such a trader were allowed to submit a sell order, he would obtain a higher expected pay-off from submitting a one tick sell limit order. Suppose that traders were restricted to submit sell orders only. The optimal sell order for a trader with valuation of 118 would be to submit a three tick sell limit order. But if such a trader were allowed to submit a buy order, he would obtain a higher expected pay-off from submitting a one tick buy limit order. The situations illustrated in Figure 5 are common enough in our sample for the model to be rejected for buy and sell order submissions jointly.

Suppose the trader selected the orders with the highest pay-offs based on Figure 5. Traders would submit one and two tick limit orders as well as market orders. The pay-offs for two tick buy and sell limit orders intersect somewhere between 116 and 117. As a consequence, all types of traders would earn positive expected pay-offs from some order submission, ignoring any order submission costs. The positive expected pay-offs are inconsistent with the predictions in

TABLE 9

Average estimated pay-offs for different order submissions

Order	Execution probability	Picking off risk	Estimated pay-off at threshold valuation
		Buy orders	
1 Tick limit	0·68	0·01	1·94
	(0·01)	(0·03)	(0·13)
2 Tick limit	0·33	−0·09	0·39
	(0·02)	(0·05)	(0·06)
3 Tick limit	0·12	−0·15	−0·02
	(0·02)	(0·10)	(0·05)
		Sell orders	
1 Tick limit	0·63	0·01	1·54
	(0·01)	(0·02)	(0·10)
2 Tick limit	0·27	0·14	0·19
	(0·02)	(0·07)	(0·09)
3 Tick limit	0·13	0·11	0·12
	(0·02)	(0·08)	(0·16)

The table reports unconditional averages of the execution probabilities, the picking off risks, and estimated pay-offs for traders with valuations equal to threshold valuations for the market and one tick, one tick and two tick, and two tick and three tick buy and sell orders. The estimated pay-offs for traders with valuations equal to the threshold valuations are computed by substituting estimates of the threshold valuations, the execution probabilities, and the picking off risks, and the order quantity into equation (11), and dividing by the order quantity. The order entry cost of c per share is set equal to zero. Asymptotic standard errors are reported in parentheses, and are computed using 50 lags.

Glosten (1994), where an equilibrium condition is that expected pay-offs are zero from submitting limit orders for traders with zero private values. Our institutional setting with discrete prices and time priority is closer to the model in Seppi (1997), where the average limit order earns a positive expected profit, and the marginal limit order earns zero profits. But Sandås (2001), also using Swedish data, rejects the zero-expected profit condition when applied only to marginal limit orders. Our results are consistent with the results in Sandås (2001).

The first column of Table 9 reports the sample averages of the execution probabilities for one, two and three tick buy and sell limit orders. The average execution probabilities show the trade-off between limit order price and execution probability. The second column of Table 9 reports the average of the estimated picking off risks. On average, buy orders away from the best bid quote and sell orders away from the best ask quote face a larger picking off risk than orders closer to the quotes. The exception is the three tick sell limit order, which has smaller picking off risk than the picking off risk for the two tick sell limit order.

The third column of Table 9 reports the average estimated pay-offs for traders with valuations equal to the threshold valuations. The order submission cost per share is set equal to zero. The estimated pay-offs are increasing the closer the order submission is to the quotes. There are two reasons for the estimated pay-offs to change across order submissions. First, the price, the execution probabilities, and the picking off risks change. Second, the estimated valuations of the trader submitting the order change. The monotonicity of the estimated pay-offs is consistent with the monotonicity of the indirect utility function in Lemma 4. The average estimated pay-offs for three tick buy limit orders are negative, although not statistically different from zero. The expected pay-offs are computed using a zero order submission cost; if there is a large order enough submission cost, three tick buy limit orders would lead to negative expected pay-offs.

5. INTERPRETING THE EVIDENCE

In our model a trader's private value measures his desire to transact. Traders with extreme private values have a strong desire to transact; they typically submit one tick limit orders or market orders. Their pay-offs are relatively insensitive to how we model monitoring costs, the possibility of multiple order submissions and resubmission strategies, since their order submissions have high execution probabilities. Such traders follow strategies similar to the pre-committed traders in Harris and Hasbrouck (1996), and their order submissions are insensitive to small changes in the execution probabilities and picking off risks. The trade-offs can rationalize the order submissions of pre-committed traders.

Traders with moderate private values are almost indifferent to trading; they typically submit two or three tick limit orders, switching between buy and sell orders. Their pay-offs are sensitive to how we model monitoring costs, the possibility of multiple order submissions and resubmission strategies, since their order submissions have low execution probabilities. Such traders follow strategies similar to the passive traders in Harris and Hasbrouck (1996), and their order submissions are sensitive to small changes in the execution probabilities and picking off risks. The trade-offs cannot rationalize the order submissions of passive traders.

We reject the model because the thresholds for the two vs. three tick buy limit order is lower than the thresholds for the two vs. three tick sell limit order. Suppose that short sales are not allowed—only owners of shares can submit sell orders. If short sales are not allowed, a trader with a low valuation who holds shares may submit a limit sell order, while a trader with the same valuation who does not hold shares may submit a buy order. We would then observe sell limit order and buy limit order submissions by traders with identical valuations, and the expected pay-off from the sell limit order submissions would be greater than the expected pay-off from the buy limit order submissions for such valuations. But in our sample, we also observe circumstances where traders place limit sell orders although the expected pay-offs from submitting limit buy orders are higher.

We reject the monotonicity restriction for the buy and sell thresholds jointly because the expected pay-offs for limit orders with low execution probabilities are too low relative to the expected pay-offs for limit orders with high execution probabilities. Modifications of the model that increase the expected pay-offs for limit orders with low execution probabilities, decrease the expected pay-offs for limit orders with high execution probabilities or a combination of the two could explain the rejection. Common fixed or variable costs of submitting orders do not change the relative expected pay-offs, and so such costs cannot explain the rejection.

The picking off risk is the covariance between changes in the common value and the order quantity that transacts. Our estimates of the common value are based on the market index. The rejection could be explained by poor estimates of the picking off risk if the estimates of the picking off risk are overestimated for limit orders with low execution probabilities relative to those for limit orders with high execution probabilities. It is possible that our estimates of the picking off risk are biased if changes in the common value are not immediately reflected by changes in the market index. On average, limit orders with low execution probabilities take longer to fill than limit orders with high execution probabilities. Perhaps the estimated changes in the common value are underestimated at shorter horizons relative to the estimated changes at longer horizons. Such a pattern could explain our rejection of the monotonicity restrictions for buy and sell thresholds jointly.

Our model does not allow for monitoring costs. In reality, traders may monitor outstanding limit orders to reduce the picking off risk they face. Monitoring costs may increase with the time that the order spends in the book. A two tick limit order has a shorter expected time to execution than a three tick limit order; such monitoring costs would lower the pay-off of three tick order relative to the pay-off of a two tick order. Such monitoring costs could not explain the rejection.

Alternatively, monitoring costs may increase with the picking off risk. A two tick sell limit order has higher picking off risk than a three tick sell limit order; such monitoring costs would lower the pay-off of a two tick sell limit order relative to the pay-off of a three tick sell limit order. Such monitoring costs could explain the rejection.

A trader's order quantity is exogenous rather than a choice variable in our model; we use quantity as a conditioning variable in our tests. Our assumption that traders choose the order price but not the quantity follows much of the theoretical literature, for example, Glosten and Milgrom (1985). According to Table 8 the model is rejected without using quantity as an instrument, suggesting that endogenous quantity is not the main reason that the model is rejected.

We compute expected pay-offs assuming that traders evaluate each order submission opportunity independently of any other order submission opportunities they may have. The monotonicity restrictions that we test do not fully characterize optimal order submissions when traders do not evaluate each order submission independently. For example, the trader may make multiple order submissions. The trader may also resubmit an order if the initial order fails to execute. The opportunity to resubmit an order is valuable only when the initial order submission is cancelled. The value of the option to resubmit would increase the pay-offs of orders with lower execution probabilities relative to the pay-offs of orders with higher execution probabilities. As a consequence, the possibility of resubmissions could explain the rejection.

The model can rationalize the order submissions of traders with extreme private values, but cannot rationalize the order submissions of traders with moderate private values. The model is rejected because the expected pay-offs from order submissions with low execution probabilities are too low relative to the expected pay-offs from limit order submissions with high execution probabilities. The rejection cannot be explained by common order submission costs or monitoring costs that increase in the time the limit order remains in the book. The rejection may be explained by our omission of monitoring costs that increase in the magnitude of the picking off risk or potential pay-offs from order resubmissions.

6. CONCLUSIONS

Most theoretical models of order submissions are based on the trade-offs between order prices, execution probabilities and picking off risks. We develop empirical restrictions of a model based on the trade-offs and compute a semiparametric test of the restrictions using a sample from the Stockholm Stock Exchange. We do not reject the restrictions for buy orders or sell orders separately and reject the restrictions for buy and sell orders jointly. The expected pay-offs from submitting limit orders away from the quotes are too low relative to the expected pay-offs from submitting orders close to the quotes to rationalize the trader's actual order submissions in our sample. The model can rationalize the order submissions of traders with extreme private values, but cannot rationalize the order submissions of traders with moderate private values.

Existing empirical studies of order submission focus on how various explanatory variables such as depth, trading activity, or time affect the traders' order submissions. But in theoretical models of order submissions, such variables enter only indirectly as determinants of the economic trade-offs between order prices, execution probabilities and picking off risks. We explicitly model the link between the economic trade-offs and the order submissions. Our model can be used to develop an ordered discrete choice model for order submissions based directly on the theory. Such a model can be used to make inferences about the private value distribution along the lines of the analysis of auctions in Elyakime, Laffont, Loisel and Vuong (1994). Estimates of the private value distribution are useful for understanding trading activity and for evaluating alternative trading rules. Hollifield, Miller, Sandås and Slive (2003) estimate such a model, accounting for the censoring that may occur when some traders find it optimal not to make any order submission.

We consider a one-shot order submission problem, with exogenous order quantity. Extending our model and empirical approach to allow for monitoring of limit orders; endogenous order quantity; multiple order submissions; and cancellations and resubmissions are all useful directions for future work.

APPENDIX A. PROOFS

Proof of Lemma 1. Assume that $u' > u$. Given $d_t^{\text{buy}*}(b, u, q) = 1$,

$$q\psi_t^{\text{buy}}(b, q)(y_t + u - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc \geq q\psi_t^{\text{buy}}(b', q)(y_t + u - p_{t,b'}^{\text{buy}}) + q\xi_t^{\text{buy}}(b', q) - qc, \qquad \text{(A.1)}$$

and $d_t^{\text{buy}*}(b', u', q) = 1$,

$$q\psi_t^{\text{buy}}(b', q)(y_t + u' - p_{t,b'}^{\text{buy}}) + q\xi_t^{\text{buy}}(b', q) - qc \geq q\psi_t^{\text{buy}}(b, q)(y_t + u' - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc. \qquad \text{(A.2)}$$

Adding inequality (A.2) to inequality (A.1), dividing by q, and rearranging

$$(\psi_t^{\text{buy}}(b, q) - \psi_t^{\text{buy}}(b', q))(u - u') \geq 0. \qquad \text{(A.3)}$$

Since $u' > u$, equation (A.3) implies that $\psi_t^{\text{buy}}(b', q) > \psi_t^{\text{buy}}(b, q)$. If the execution probability is monotone in distance from the best ask quote, then equation (A.3) implies that $b' \leq b$. The proof for the sell side is symmetric. ‖

Proof of Lemma 2. If it is optimal for a trader with private value u to submit a buy order, then it is also optimal for traders with private values $u' > u$ to submit buy orders. Let s be an arbitrary sell order, and suppose that $d_t^{\text{buy}*}(b, u, q) = 1$. After dividing by q,

$$\begin{aligned}
\psi_t^{\text{buy}}(b, q)(y_t + u' - p_{t,b}^{\text{buy}}) + \xi_t^{\text{buy}}(b, q) - c &> \psi_t^{\text{buy}}(b, q)(y_t + u - p_{t,b}^{\text{buy}}) + \xi_t^{\text{buy}}(b, q) - c \\
&\geq \psi_t^{\text{sell}}(s, q)(p_{t,s}^{\text{sell}} - y_t - u) - \xi_t^{\text{sell}}(s, q) - c \\
&\geq \psi_t^{\text{sell}}(s, q)(p_{t,s}^{\text{sell}} - y_t - u') - \xi_t^{\text{sell}}(s, q) - c. \qquad \text{(A.4)}
\end{aligned}$$

The first line follows because $u' > u$; the second line follows because it is optimal for a trader with private value u to submit a buy order at b; the third line follows because $u' > u$. Symmetric arguments hold for sellers. Thus, there exists $u_t^{\text{buy}} \geq u_t^{\text{sell}}$ such that all traders with private values $u > u_t^{\text{buy}}$ optimally submit buy orders, and all traders with private values $u < u_t^{\text{sell}}$ optimally submit sell orders. Monotonicity of the associated thresholds follows from Lemma 1. ‖

Proof of Lemma 3. The result follows from Lemmas 1 and 2. ‖

Proof of Lemma 4. 1. $V_t(u_t, q) \geq 0$, since the trader can always submit no order and earn a zero pay-off. To show convexity, consider two private values, u and u'. Let $0 < \lambda < 1$ and let $u_\lambda = \lambda u + (1 - \lambda)u'$. Suppose $d_t^{\text{buy}*}(b, u_\lambda, q) = 1$,

$$V_t(y_t + u, q) \geq q\psi_t^{\text{buy}}(b, q)(y_t + u - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc, \qquad \text{(A.5)}$$

and

$$V_t(y_t + u', q) \geq q\psi_t^{\text{buy}}(b, q)(y_t + u' - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc. \qquad \text{(A.6)}$$

Taking the convex combination of inequalities (A.5) and (A.6) and using the definition of u_λ,

$$\begin{aligned}
\lambda V_t(y_t + u, q) + (1 - \lambda)V_t(y_t + u', q) &\geq q\psi_t^{\text{buy}}(b, q)(y_t + u_\lambda - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc \\
&= V_t(y_t + u_\lambda, q). \qquad \text{(A.7)}
\end{aligned}$$

The proof is similar if it is optimal for the trader with private value u_λ to submit a sell order or to submit no order.

2. Let $d_t^{\text{buy}*}(b, u, q) = 1$ and $u' > u$.

$$\begin{aligned} V_t(y_t + u', q) &\geq q\psi_t^{\text{buy}}(b, q)(y_t + u' - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc \\ &> q\psi_t^{\text{buy}}(b, q)(y_t + u - p_{t,b}^{\text{buy}}) + q\xi_t^{\text{buy}}(b, q) - qc \\ &= V_t(y_t + u, q). \end{aligned} \tag{A.8}$$

The first line follows from the definition of $V_t(y_t + u', q)$; the second line follows since $u' > u$; and the third line follows since $d_t^{\text{buy}*}(b, u, q) = 1$.

3. The proof for sellers is symmetric to that in part 2 above. ‖

APPENDIX B. COINTEGRATION RESULTS

We assume that there is a strictly positive tick size; feasible order prices are elements of a countable set. Order the set of feasible prices from lowest to highest, so that $P_i < P_{i+1}$. Let $Q_{ij,t}$ be the order quantity outstanding at the i-th price at time t, submitted at time $t - j$, with $Q_{ij,t} > 0$ denoting buy quantities, $Q_{ij,t} < 0$ denoting sell quantities, and $Q_{ij,t} = 0$ denoting that no order quantity is outstanding.

The rules of the trading mechanism imply that there cannot be both buy and sell orders outstanding at the same price at the same time.

Let the common value at time t equal y_t. Define the feasible relative prices at time t as the elements of the set of feasible prices minus y_t. The relative order book at t is

$$H_t = (P_i - y_t, Q_{ij,t}) \qquad \text{for } i = 1, 2, 3, \dots, \infty, \qquad \text{and} \qquad j = 1, 2, 3, \dots, \infty. \tag{B.1}$$

We make the following assumptions.

- CI1 The maximum life of each limit order is some finite integer $\Upsilon < \infty$.
- CI2 Suppose a limit order is submitted to the limit order book at time t. The conditional probability that the order is cancelled at time $t + \tau$ for $\tau < \Upsilon$ depends on a finite-dimensional vector of variables, $R_{t+\tau}$. The conditional probabilities are uniformly bounded below by a strictly positive constant.
- CI3 The process (R_t, δ_t) is a Markov process and satisfies Condition M of Stokey, Lucas and Prescott (1989, p. 348).
- CI4 The conditional distribution of δ_t, u_t, q_t only depends on R_t, and δ_t, u_t, q_t are conditionally independent.
- CI5 The random variables δ_t, u_t, q_t each have uniformly bounded support.
- CI6 A trader at t only conditions his order submissions on the order book relative to the common value, H_t, and R_t.
- CI7 The cost per share of entering orders, c, is strictly positive.

Lemma B1. *Under assumptions CI1–CI7, H_t is characterized by a finite number of elements, each of which is a bounded random variable.*

Proof of Lemma B1. We show that at most a finite number of relative prices have order quantities at any time. Let $[\underline{u}, \overline{u}]$ be the support for the private value and let $[\underline{\delta}, \overline{\delta}]$ be the support for common value innovations. Let p_t^{buy} be the price of a buy order submission at time t, and p_t^{sell} the price of a sell order submission at t.

A buyer never submits an order that leads to a negative surplus with probability one, and the highest possible valuation that a buyer could have is his valuation at the time the order fills, which is bounded by assumption. Therefore,

$$\begin{aligned} p_t^{\text{buy}} &\leq y_{t+\Upsilon} + \overline{u} \\ &\leq y_t + \Upsilon\overline{\delta} + \overline{u}. \end{aligned} \tag{B.2}$$

Similarly for sellers,

$$p_t^{\text{sell}} \geq y_t + \Upsilon\underline{\delta} + \underline{u}. \tag{B.3}$$

With a positive cost per share for order entry, no buyer would ever submit an order that has a zero execution probability. Since a limit buy order only transacts with a future sell order and the longest that a limit order lasts is Υ, for a buy order to have a positive probability of being filled, it must satisfy

$$\begin{aligned} p_t^{\text{buy}} &\geq y_{t+\Upsilon} + \Upsilon\underline{\delta} + \underline{u} \\ &\geq y_t + \Upsilon\underline{\delta} + \Upsilon\underline{\delta} + \underline{u} \\ &= y_t + 2\Upsilon\underline{\delta} + \underline{u}. \end{aligned} \tag{B.4}$$

Combining inequalities (B.2) and (B.4),

$$y_t + 2\Upsilon\underline{\delta} + \underline{u} \leq p_t^{\text{buy}} \leq y_t + \Upsilon\overline{\delta} + \overline{u}, \tag{B.5}$$

or

$$2\Upsilon\underline{\delta} + \underline{u} \leq p_t^{\text{buy}} - y_t \leq \Upsilon\overline{\delta} + \overline{u}. \tag{B.6}$$

A similar result holds for sell orders. The relative prices in the relative order book are all bounded at the time of entry:

$$2\Upsilon\underline{\delta} + \underline{u} \leq p_t - y_t \leq \Upsilon 2\overline{\delta} + \overline{u}, \tag{B.7}$$

with p_t the order price submitted at t.

Since orders last for up to Υ periods, there can be at most Υ orders outstanding at any time, and so the relative prices of all orders in the relative order book are bounded. By assumption, order quantity is bounded. ‖

From Lemma B1, there are a finite number of orders outstanding at any time with bounded relative prices. Let that finite number be $\mathcal{M}$. The relative order book can be represented by

$$H_t = (P_{i,t} - y_t, Q_{ij,t}) \text{ for } i = 1, 2, \ldots, \mathcal{M}, \qquad \text{and} \qquad j = 1, 2, 3, \ldots, \Upsilon, \tag{B.8}$$

with $P_{i,t} < P_{i+1,t}$ and $|P_{i,t} - y_t|$ strictly bounded by a finite constant for $i = 1, 2, \ldots, \mathcal{M}$.

Lemma B2. *Under assumptions CI1–CI7,* (H_t, R_t, δ_t) *is a stationary Markov process, with a unique ergodic set.*

Proof of Lemma B2. By assumption, new order submissions depend upon the relative order book, H_t, the state vector R_t, and the trader's private value. The conditional distribution of H_{t+1} depends upon H_t, new order submissions, cancellations and innovations in the common value. By assumption, (R_t, δ_t) is Markov, and the distribution of q_t depends only on R_t. Therefore, the process (H_t, R_t, δ_t) is also Markov.

The hazard rates for cancellation are bounded below by a strictly positive number; there is a strictly positive probability that all orders will cancel from any state, leaving all the quantities in the book zero. The conditional distribution of relative prices depends δ_t. By assumption, (R_t, δ_t) satisfies Condition M. Therefore, (H_t, R_t, δ_t) also satisfies Condition M. Theorem 11.12 of Stokey *et al*. (1989) then applies and so (H_t, R_t, δ_t) is a stationary process with unique ergodic set. ‖

Define the random variable

$$\dot{\varepsilon}_t = \begin{cases} \max_{i=1,\ldots,\mathcal{M}}\left\{ P_{i,t} - y_t \,\middle|\, \sum_j Q_{ij,t} > 0 \right\} & \text{if some } Q_{ij,t} > 0, \\ 0 & \text{else.} \end{cases} \tag{B.9}$$

The random variable $\dot{\varepsilon}_t$ is the difference between the best bid quote and the common value, if there are buy orders in the relative book, or zero if there are no buy orders in the book.

Lemma B3. *Under assumptions CI1–CI7,* $\dot{\varepsilon}_t$ *is stationary.*

Proof of Lemma B3. The random variable $\dot{\varepsilon}_t$ is a mapping of (H_t, R_t, δ_t) to $\mathcal{R}$. Under assumptions CI1–CI7 (H_t, R_t, δ_t) is a stationary Markov process with a unique ergodic set. ‖

The best bid quote does not exist if there are no buy orders in the relative book. The ergodic set for the relative order book contains the states where there are no orders in the book. The next assumption guarantees that the ergodic set also contains books with limit buy orders in the book.

CI8 Suppose that there are no orders in the book at time t. Then, the probability that a buy order is submitted is uniformly strictly positive, for all possible values of R_t.

Lemma B4. *Under assumptions CI1–CI8, the best bid quote and the common value are cointegrated.*

Proof of Lemma B4. By assumption, y_t is $I(1)$. From Lemma B3, $\dot{\varepsilon}_t$ is stationary and ergodic and is equal to the difference between the best bid quote and y_t when there are buy orders in the relative book. The ergodic set contains books with no orders. Assumption CI8 implies that books with buy orders are also in the ergodic set; and that the indicator function for the event of buy orders in the book is also stationary and ergodic. The process $\varepsilon_{t'}$ formed by sampling the process $\dot{\varepsilon}_t$ when there are buy orders on the book is also stationary. ‖

APPENDIX C. ECONOMETRICS APPENDIX

Our data consist of observations of the vector of M conditioning variables, X_t, the decision indicators, $d^{\text{sell}}_{t,\ddot{s}}$, for $\ddot{s} \in \ddot{S}(X_t)$, $d^{\text{buy}}_{t,\ddot{b}}$, for $\ddot{b} \in \ddot{B}(X_t)$, the realized fills for each order, and the realized product of the fills and the changes in the estimated common value for each order. Let w_t be the vector of variables whose conditional expectations we compute. Define the conditional expectation functions

$$C^{\text{sell}}_{\ddot{s}}(X_t) \equiv E[w|d^{\text{sell}}_{\ddot{s},t} = 1, X_t], \tag{C.1}$$

$$C^{\text{buy}}_{\ddot{b}}(X_t) \equiv E[w|d^{\text{buy}}_{\ddot{b},t} = 1, X_t] \tag{C.2}$$

for $\forall\, \ddot{s} \in \ddot{S}(X_t)$ and $\forall\, \ddot{b} \in \ddot{B}(X_t)$, and define the vector of conditional expectations

$$C(X_t) \equiv (C^{\text{sell}}_{\ddot{s}_1}(X_t), C^{\text{sell}}_{\ddot{s}_2}(X_t), \ldots, C^{\text{buy}}_{\ddot{b}_2}(X_t), C^{\text{buy}}_{\ddot{b}_1}(X_t)). \tag{C.3}$$

The object to be estimated depends on the vector valued function $\rho(C(X_t), X_t)$. Define

$$\varrho \equiv E[I(X_t \in \bar{X})\rho(C(X_t), X_t)], \tag{C.4}$$

where $I(X_t \in \bar{X})$ is a trimming indicator for the set $\bar{X}$ in the interior of the support of X_t. Our estimator for ϱ is

$$\hat{\varrho}_T \equiv \frac{1}{T}\sum\nolimits_{t=1}^{T} I(X_t \in \bar{X})\rho(\hat{C}(X_t), X_t), \tag{C.5}$$

where $\hat{C}(X_t)$ is estimated using a non-parametric kernel regression. For example,

$$\hat{C}^{\text{sell}}_{\ddot{s}}(X_t) \equiv \frac{\sum\nolimits_{t'\neq t}^{T} w_{t'} d^{\text{sell}}_{\ddot{s},t'} \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}{\sum\nolimits_{t'\neq t}^{T} \mathcal{K}(h_T^{-1}(X_{t'} - X_t))}, \tag{C.6}$$

where h_T is a bandwidth and $\mathcal{K}$ is a multidimensional kernel function.

In our applications, the vector of conditional expectations is

$$C(X_t) \equiv (\psi^{\text{sell}}(\ddot{s}_1, X_t), \xi^{\text{sell}}(\ddot{s}_1, X_t), \ldots, \psi^{\text{buy}}(\ddot{b}_1, X_t), \xi^{\text{buy}}(\ddot{b}_1, X_t)). \tag{C.7}$$

For testing monotonicity of the execution probabilities,

$$\rho(C(X_t), X_t) \equiv \begin{pmatrix} \psi^{\text{buy}}(\ddot{b}_1, X_t) & - & \psi^{\text{buy}}(\ddot{b}_2, X_t), \\ \psi^{\text{buy}}(\ddot{b}_2, X_t) & - & \psi^{\text{buy}}(\ddot{b}_3, X_t), \\ & \vdots & \\ \psi^{\text{sell}}(\ddot{s}_2, X_t) & - & \psi^{\text{sell}}(\ddot{s}_3, X_t) \\ \psi^{\text{sell}}(\ddot{s}_1, X_t) & - & \psi^{\text{sell}}(\ddot{s}_2, X_t) \end{pmatrix} \otimes z_t^{++}, \tag{C.8}$$

where z_t^{++} are strictly positive measurable functions of the vector X_t, and $\otimes$ is the Kronecker product. For testing monotonicity of the thresholds, define the composite function

$$\rho(\theta(C(X_t), X_t), X_t) \equiv \begin{pmatrix} \theta^{\text{buy}}(\ddot{b}_1, \ddot{b}_2, X_t) & - & \theta^{\text{buy}}(\ddot{b}_2, \ddot{b}_3, X_t) \\ \theta^{\text{buy}}(\ddot{b}_2, \ddot{b}_3, X_t) & - & \theta^{\text{buy}}(\ddot{b}_3, \ddot{b}_4, X_t) \\ & \vdots & \\ \theta^{\text{sell}}(\ddot{s}_3, \ddot{s}_4, X_t) & - & \theta^{\text{sell}}(\ddot{s}_2, \ddot{s}_3, X_t) \\ \theta^{\text{sell}}(\ddot{s}_2, \ddot{s}_3, X_t) & - & \theta^{\text{sell}}(\ddot{s}_1, \ddot{s}_2, X_t) \end{pmatrix} \otimes z_t^{++}, \tag{C.9}$$

and

$$\theta(C(X_t), X_t) \equiv (\theta^{\text{buy}}(\ddot{b}_1, \ddot{b}_2, X_t; C(X_t)), \ldots, \theta^{\text{sell}}(\ddot{s}_1, \ddot{s}_2, X_t; C(X_t))), \tag{C.10}$$

with

$$\theta^{\text{sell}}(\ddot{s}_1, \ddot{s}_2, X_t; C(X_t)) = p_{\ddot{s}_1,t} - \frac{(p_{\ddot{s}_2,t} - p_{\ddot{s}_1,t})\psi^{\text{sell}}(\ddot{s}_2, X_t) + (\xi^{\text{sell}}(\ddot{s}_1, X_t) - \xi^{\text{sell}}(\ddot{s}_2, X_t))}{\psi^{\text{sell}}(\ddot{s}_1, X_t) - \psi^{\text{sell}}(\ddot{s}_2, X_t)}. \tag{C.11}$$

Under the assumptions EC1–EC6, the results in Robinson (1989) and Ahn and Manski (1993)[5] imply that $\sqrt{T}(\hat{\varrho}_T - \varrho)$ converges in distribution to a normal random vector with mean zero and covariance matrix defined in equation (C.24) in what follows.

EC1 $X_t, w_t, d^{\text{sell}}_{\ddot{s},t}, d^{\text{buy}}_{\ddot{b},t}$, are absolutely regular and the beta-mixing coefficient is $o(j^{-\nu})$. Also,

$$\sup_{X\in\bar{X}} \|\rho(C(X), X)\|^{\varphi} < \infty, \tag{C.12}$$

where $\nu > 1 + \frac{2}{\varphi-2}$. For a definition of absolute regularity and the beta-mixing coefficient, see Robinson (1989).

EC2 (a) The distribution of X_t has Lebesgue density π that is bounded and at least $M+1$ times differentiable, with the first $M+1$ derivatives bounded.

(b) The realized fills and picking off risks have bounded support.

(c) $C(X_t)$ is $M+1$ times differentiable with bounded derivatives.

(d) The conditional choice probabilities,

$$\alpha^{\text{sell}}_{\ddot{s}}(X_t) = \text{Prob}(d^{\text{sell}}_{\ddot{s}} = 1 \mid X_t) \tag{C.13}$$

are $M+1$ times differentiable with bounded derivatives for all $\ddot{s} \in \ddot{S}(X_t)$ and similarly on the buy side. The function $\pi(X)\alpha^{\text{sell}}_{\ddot{s}}(X)$ satisfies

$$\inf_{X\in\bar{X}} \pi(X)\alpha^{\text{sell}}_{\ddot{s}}(X) > 0 \tag{C.14}$$

for $\forall\, \ddot{s} \in \ddot{S}(X_t)$, and similarly for the buy side.

EC3 (a) The partial derivatives satisfy

$$\sup_{X_t\in\bar{X}} \left\| \frac{\partial\rho(C(X_t), X_t)}{\partial C(X_t)} \right\| < \infty. \tag{C.15}$$

(b) There is an $F < \infty$ such that the cross partial derivatives satisfy

$$\sup_{X_t\in\bar{X}} \left\| \frac{\partial^2\rho(C(X_t), X_t)}{\partial C(X_t)\partial C(X_t)'} \right\| < F. \tag{C.16}$$

EC4 Define the matrix of expected derivatives as

$$\mu(X) \equiv E\left[\left. \frac{\partial\rho(C(X), X)}{\partial C(X)} \right| X \right], \tag{C.17}$$

with generic element $\mu_{ij}(X)$ with $M+1$ bounded derivatives satisfying

$$\frac{\mu_{ij}(X)}{\alpha^{\text{sell}}_{\ddot{s}}(X)} < \infty. \tag{C.18}$$

EC5 Define the vector of error terms

$$\epsilon^{\text{sell}}_{\ddot{s},t} = d^{\text{sell}}_{\ddot{s},t}[w_t - C^{\text{sell}}_{\ddot{s}}(X_t)], \tag{C.19}$$

with a similar definition for $\epsilon^{\text{buy}}_{\ddot{b},t}$, and

$$\epsilon_t = \sum\nolimits_{\ddot{s}\in\ddot{S}} \epsilon^{\text{sell}}_{\ddot{s},t} + \sum\nolimits_{\ddot{b}\in\ddot{B}} \epsilon^{\text{buy}}_{\ddot{b},t}. \tag{C.20}$$

There exists a positive semi-definite matrix $\mathcal{C}$ such that

$$\sup_{X_t\in\bar{X}} \lim_{LL\to\infty} \sum\nolimits_{ll=-LL}^{LL} E[\epsilon_{t-ll}\epsilon'_{t+ll}|X_t] < \mathcal{C}. \tag{C.21}$$

EC6 (a) The bandwidth sequence is such that $Th_T^{2(M+1)} \to \infty$, $T^{1-2\kappa}h_T^{2M} \to 0$ as $T \to \infty$ for some $\kappa > 0$.

(b) The kernel function $\mathcal{K}$ is bounded and symmetric around zero, satisfying $\int \mathcal{K}(z)dz = 1$ and $\int |z|^{2(M+1)}\mathcal{K}(z)dz < \infty$. There exists $\gamma > 0$ and $c < \infty$ such that $\mathcal{K}$ satisfies the Lipschitz condition that $|\mathcal{K}(z) - \mathcal{K}(z')| \le c|z - z'|^{\gamma}$ for all $z, z' \in \mathcal{R}^M$.

(c) The first M moments of $\mathcal{K}$ are zero.

5. Ahn and Manski (1993) consider i.i.d. data. The uniform consistency results from Collomb and Härdle (1986) regarding the kernel estimators applied in Ahn and Manski (1993) continue to apply under our assumptions.

Define

$$\eta_t = \rho(C(X_t), X_t) - \varrho \tag{C.22}$$

and the vector

$$e_t \equiv \left(\frac{\epsilon^{\text{sell}}_{\ddot{s}_1,t}}{\alpha^{\text{sell}}_{\ddot{s}_1,t}(X_t)}, \dots, \frac{\epsilon^{\text{sell}}_{\ddot{s}_K,t}}{\alpha^{\text{sell}}_{\ddot{s}_K,t}(X_t)}, \frac{\epsilon^{\text{buy}}_{\ddot{b}_1,t}}{\alpha^{\text{buy}}_{\ddot{b},t}(X_t)}, \dots, \frac{\epsilon^{\text{buy}}_{\ddot{b}_L,t}}{\alpha^{\text{buy}}_{\ddot{b}_L,t}(X_t)} \right). \tag{C.23}$$

Then,

$$\mathcal{A} = \lim_{LL\to\infty} \sum_{ll=-LL}^{LL} E[(\eta_{t-ll} + \mu(X_{t-ll})' e_{t-ll})(\eta_{t+ll} + \mu(X_{t+ll})' e_{t+ll})']. \tag{C.24}$$

We estimate η_t with $\rho(\hat{C}(X_t), X_t) - \hat{\varrho}_T$. We estimate e_t using the kernel estimators in equations (C.19) and (C.20) and using kernel estimators for the conditional choice probabilities in equation (C.13). We use a Newey and West (1987) procedure to form an estimator for $\mathcal{A}$.

The thresholds are linear in β, and so the super-consistency of the cointegrating regression implies that the asymptotic distribution is unaffected by pre-estimating β. See De Jong (2001) for details.

Following Altug and Miller (1998) and Hotz and Miller (1993), and the simulation evidence in Robinson (1989, pp. 521–522) we use independent Gaussian product kernels in forming estimates of the conditional expectations. We use bandwidths equal to

$$6 \times 1{\cdot}06 \times \hat{\sigma}(X_{it}) T^{\frac{1}{2\times 6+2}}. \tag{C.25}$$

Here, $X_t = (X_{1t}, \dots, X_{5t})$ are the conditioning variables, with $\hat{\sigma}(X_{it})$ the associated sample standard deviations. We trim the outer 5% of the observations:

$$\bar{X} = \{X|(X - \widehat{E}(X_t))\widehat{\text{cov}}(X_t)^{-1}(X - \widehat{E}(X_t))' \leq 0{\cdot}95\}, \tag{C.26}$$

where $\widehat{\text{cov}}(X_t)$ is the sample covariance matrix of the conditioning information and $\widehat{E}(X_t)$ is the sample mean, leaving us with 19,732 observations.

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