CEMMAP Lectures on Contracts & Market Microstructure

4. Limit Order Markets

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What are Limit Order Markets?

Market microstructure

- Markets exist because individuals benefit from voluntary exchange.
- Competitive equilibrium is a useful modeling tool to parsimoniously capture the fundamentals of trade.
- But can we replace the fiction of a Walrasian auctioneer setting prices with models based on:
  - institutions, or trading rules, designed to facilitate trade
  - where behavior can be modeled as a noncooperative game.
- Focusing on one such institution, three questions frame this lecture:

1. What is a *limit order market* (LOM)?
2. Do LOM models have empirical content?
   - Can LOM models be tested (falsified)?
   - Note that empirical content does not imply identification.
3. How efficient are LOMs in allocating resources?

What is an LOM?

The order book

- The trading mechanism for a given security in a generic limit order market can be described by:
  1. the *order book*.
  2. the rules and procedures for submitting and withdrawing orders.
- At any given instant during business hours, there is:
  1. a list of unfilled orders to buy the security
  2. another list of unfilled orders to sell the security
- Each *limit order* on each list consists of:
  1. a price
  2. a quantity
  3. a submission time
- Every order on the sell list is marked with a higher price than every order on the buy list.
- The difference between the lowest unfilled sell order (the *ask*) and the highest unfilled buy order (the *bid*) is called the *spread*.

What is an LOM?

Orders

- An investor seeking to trade the security in this market can:
  1. add to one of the lists by placing a buy (sell) order, which is lower than the offer (higher than the bid). This is called making a limit buy (sell) order.
  2. execute a trade by accepting the ask (bid) on the other side of the market. This is called a *market buy (sell) order*.
- If two unfilled orders have the same price, then the order submitted earlier is executed first.
- Investors wishing to execute only a proportion of another investor's unfilled limit order with their own market order may do so.
- Investors wishing to withdraw their limit orders may so at any before a market order cancels them with a transaction.
- Summarizing limit order markets exhibit *price/time precedence*.

What is an LOM?

Trading window

Price(Markup)	Quantity (Cum.)	Revenue\Cost (Cum.)	Duration
6000.00 (-291.24)	4 (4)	24000.00 (24000.00)	59603
5800.00 (-491.24)	9 (13)	52200.00 (76200.00)	59634
3800.00 (2491.24)	2 (2)	7600.00 (7600.00)	59301
200.00 (6091.24)	4 (6)	800.00 (8400.00)	59197

What is an LOM?

Data on limit order markets

- A limit order market (LOM) for financial securities offers an **excellent laboratory analyzing trading mechanisms** where there are many players on both sides of the market:
  1. **Transparent rules** govern trading in limit order markets, easily modeled (*compared to labor markets and transactions in industrial organization*).
  2. Different units of the securities are **perfect substitutes** and hece comparable (*in contrast to many real assets*).
  3. The volume and **value of traded securities is huge**, inducing traders to perform as well as they can (*unlike experimental settings*).
  4. The exchanges collects **reliable data** because they undergird contracts between transacting parties (*relative to say survey data or information small businesses provide to the government for taxation purposes*).
- One deficiency: researchers cannot (usually) observe the the **wealth portfolio** of the source of the orders.

An LOM Model (Hollifield, Miller and Sandas, 2004)

Valuation

- At time $t \in \{1, 2, \ldots\}$ just one trader has his only opportunity to submit an order for one (or more generally exogenously determined) unit(s) of an asset.
- Trader t is risk neutral and values the unit at:

$$v_t = u_t + y_t$$

  where:
  - u_t is independent and identically distributed with support on the real line and probability distribution function $G(u)$.
  - y_t is a Martingale, meaning $E_t[y_{t+1}] = y_t$.
- We interpret y_t as the expected liquidation value of an asset that pays no dividends in the meantime.
- Trader t observes both components.

An LOM Model

Prices

- Traders buy and sell on a discrete price grid $\{\ldots, p_{j-1}, p_j, p_{j+1}, \ldots\}$.
- The difference $p_{j+1} - p_j$ is called the tick size.
- Denote by $\left\{ p_{0t}^{(b)}, p_{1t}^{(b)} \ldots \right\}$ the buy prices trader t can choose from:
  - $p_{0t}^{(b)}$ is the lowest limit order sell offer (the ask price).
  - Trader t submits a market buy order by selecting $p_{0t}^{(b)}$.
  - $p_{kt}^{(b)}$ is k ticks below $p_{0t}^{(b)}$.
  - Trader t submits a limit buy order by selecting $p \in \left\{ p_{1t}^{(b)}, p_{2t}^{(b)} \ldots \right\}$.
- Similarly $\left\{ p_{0t}^{(s)}, p_{1t}^{(s)} \ldots \right\}$ are sell prices, and trader t can submit a:
  - market sell order by selecting the (highest limit order) bid price $p_{0t}^{(s)}$.
  - limit sell order $p_{kt}^{(s)}$ that is k ticks above $p_{0t}^{(s)}$.
- The difference $p_{0t}^{(b)} - p_{0t}^{(s)}$ is called the spread (ask price less bid price).
- Prices on and inside the spread can be selected by a buyer or seller.

- Let $d_{kt}^{(b)} \in \{0, 1\}$ and $d_{kt}^{(s)} \in \{0, 1\}$ for $k \in \{0, 1, 2, \ldots\}$, where $d_{kt}^{(b)} = 1$ means t submits a buy order at price $p_{kt}^{(b)}$.
- Assume t submits at most one order, implying:

$$\sum_{k=0}^{\infty} \left(d_{kt}^{(b)} + d_{kt}^{(s)} \right) \leq 1$$

- Suppose $d_{kt}^{(s)} = 1$ and let:
  - $r_{k,t,t+\tau}^{(s)} = 1$ if the order is *cancelled* $t + \tau$ (otherwise $r_{k,t,t+\tau}^{(s)} = 0$).
  - $q_{k,t,t+\tau}^{(s)} = 1$ if the order is *filled* at $t + \tau$ (otherwise $q_{k,t,t+\tau}^{(s)} = 0$).
  - $r_{k,t,t+\tau}^{(b)}$ and $q_{k,t,t+\tau}^{(b)}$ be similarly defined.

- Assume $r^{(s)}_{k,t,t+\tau}$ and $r^{(b)}_{k,t,t+\tau}$ are independent exogenous processes.
- Execution is endogenous (because another trader is involved):
  - $q^{(s)}_{0,t,t} = 1$ because market orders execute immediately.
  - If $q^{(s)}_{k,t,t+\tau} = 1$ for some $\tau > 0$ then $d^{(b)}_{0,t+\tau} = 1$. (Every trade fills a limit order with a market order.)
  - Price precedence implies if $q^{(s)}_{k,t,t+\tau} = 1$ and $d^{(s)}_{k't'} = 1$ for some $k' < k$ and $t' < t + \tau$ then $q^{(s)}_{k',t',\rho} = 1$ or $r^{(s)}_{k',t',\rho} = 1$ for some $\rho \leq t + \tau$.
- Trader t chooses $d_t \equiv \left(d^{(b)}_{0t}, d^{(s)}_{0t}, d^{(b)}_{1t}, d^{(s)}_{1t}, \ldots \right)$ to maximize:

$$E_t \left\{ \sum_{k=0}^{\infty} \sum_{\tau=0}^{\infty} \left[\begin{array}{c} d^{(b)}_{kt} \prod_{\rho=0}^{\tau} \left(1 - r^{(b)}_{kt,t+\rho}\right) q^{(b)}_{kt,t+\tau} \left(v_{t+\tau} - p^{(b)}_{kt}\right) \\ +d^{(s)}_{kt} \prod_{\rho=0}^{\tau} \left(1 - r^{(s)}_{kt,t+\rho}\right) q^{(s)}_{kt,t+\tau} \left(p^{(s)}_{kt} - v_{t+\tau}\right) \end{array} \right] \right\}$$

Equilibrium

Existence and uniqueness

- This is (isomorphic to) a *perfect information game*:
  - Each trader t observes the value of y_t and all the outstanding limit orders (comprising the limit order book).
  - Traders move sequentially each trader is fully informed about the moves of previous agents.
- If the game has a finite horizon, then it is straightforward to establish that (generically) a unique equilibrium exists.
- Let $\widehat{d}_{kt}^{(b)}$ and $\widehat{d}_{kt}^{(s)}$ denote equilibrium $d_{kt}^{(b)}$ and $d_{kt}^{(s)}$ choices.
- Similarly let $\widehat{q}_{kt}^{(b)}$ and $\widehat{q}_{kt}^{(s)}$ denote equilibrium $q_{kt}^{(b)}$ and $q_{kt}^{(s)}$ executions.

Equilibrium

Conditional choice probabilities, execution probabilities and picking off risks

- To characterize the equilibrium choices, define:
  - *conditional choice probabilities of submission*:

$$\lambda_{kt}^{(b)} \equiv \int \widehat{d}_{kt}^{(b)} dG\left(u\right)$$

  - *execution probabilities*:

$$\psi_{kt}^{(b)} \equiv E_t \left[\sum_{\tau=0}^{\infty} \widehat{q}_{kt,t+\tau}^{(b)} \prod_{\rho=1}^{\tau} \left(1 - r_{kt,t+\rho}^{(b)}\right) \right]$$

  - *picking-off risk*:

$$\xi_{kt}^{(b)} \equiv E_t \left[\sum_{k=0}^{\infty} \left(y_{t+\tau} - y_t\right) \widehat{q}_{kt,t+\tau}^{(s)} \right]$$

- Thus trader t chooses d_t to maximize:

$$\sum_{k=0}^{\infty} \left\{ d_{kt}^{(b)} \left[\psi_{kt}^{(b)} \left(v_t - p_{kt}^{(b)} \right) + \xi_{kt}^{(b)} \right] + d_{kt}^{(s)} \left[\psi_{kt}^{(s)} \left(p_{kt}^{(s)} - v_t \right) - \xi_{kt}^{(s)} \right] \right\}$$

An LOM Model

A revealed preference argument (Lemma 1, HMS 2004)

- Suppose $v_t = u + y_t$ and $v'_t = u' + y_t$.
- If $d_{kt}^{(b)}(u) = d_{k't}^{(b)}(u') = 1$, then:

$$\begin{aligned}
\psi_{kt}^{(b)}\left(v_t - p_{kt}^{(b)}\right) + \xi_{kt}^{(b)} - c &\geq \psi_{k't}^{(b)}\left(v_t - p_{k't}^{(b)}\right) + \xi_{k't}^{(b)} - c \\
\psi_{k't}^{(b)}\left(v'_t - p_{k't}^{(b)}\right) + \xi_{k't}^{(b)} - c &\geq \psi_{kt}^{(b)}\left(v'_t - p_{kt}^{(b)}\right) + \xi_{kt}^{(b)} - c
\end{aligned}$$

- Add the inequalities together; then add to both sides:

$$\psi_{kt}^{(b)} p_{kt}^{(b)} + \psi_{k't}^{(b)} p_{k't}^{(b)} + \left[\psi_{kt}^{(b)} + \psi_{k't}^{(b)}\right] y_t + 2c - \xi_{kt}^{(b)} - \xi_{k't}^{(b)}$$

- Rearrange the resulting inequality to yield:

$$\left[\psi_{kt}^{(b)} - \psi_{k't}^{(b)}\right]\left(v_t - v'_t\right) = \left[\psi_{kt}^{(b)} - \psi_{k't}^{(b)}\right]\left(u - u'\right) \geq 0$$

- For example if $u \geq u'$ then $\psi_{kt}^{(b)} \geq \psi_{k't}^{(b)}$.
- Since $\psi_{kt}^{(b)}$ is decreasing in k, in this case $p_{k't}^{(b)} \leq p_{kt}^{(b)}$.
- An analogous result holds for the sell side.

An LOM Model

Threshold valuations

- The empirical content of LOM models can be derived from a monotonicity property of *threshold valuations*.
- Define $\theta_t^{(b)}(k,k')$ as the valuation of a trader indifferent between submitting $p_{kt}^{(b)}$ versus $p_{k't}^{(b)}$:

$$\begin{aligned} & \psi_{kt}^{(b)}\left(\theta_t^{(b)}(k,k') + y_t - p_{kt}^{(b)}\right) + \xi_{kt}^{(b)} - c \\ = \; & \psi_{k't}^{(b)}\left(\theta_t^{(b)}(k,k') + y_t - p_{k't}^{(b)}\right) + \xi_{k't}^{(b)} - c \\ \Rightarrow \theta_t^{(b)}(k,k') = \; & p_{kt}^{(b)} + \frac{\left[p_{kt}^{(b)} - p_{k't}^{(b)}\right]\psi_{k't}^{(b)} + \xi_{k't}^{(b)} - \xi_{kt}^{(b)}}{\psi_{k't}^{(b)} - \psi_{kt}^{(b)}} \end{aligned}$$

- Similar expressions can be defined for traders indifferent between:
  - selling at two different prices
  - buying a unit versus selling a unit (at a higher price)
  - trading at some price versus not trading at all.

An LOM Model

Monotonicity of threshold valuations (Lemmas 2 and 3, HMS 2004)

- The revealed preference argument implies:

$$\theta_t^{(b)}(k, k+1) > \theta_t^{(b)}(k+1, k+2)$$

- An analogous argument applies to the sell side:

$$\theta_t^{(s)}(k, k+1) < \theta_t^{(b)}(k+1, k+2)$$

- Using similar reasoning we can show:

$$\theta_t^{(b)}(0, 1) < \theta_t^{(s)}(0, 1)$$

- These inequalities characterize the equilibrium submission strategy.

An LOM Model

Illustrating the indirect utility function (HMS 2004, Figure 2, page 1039)

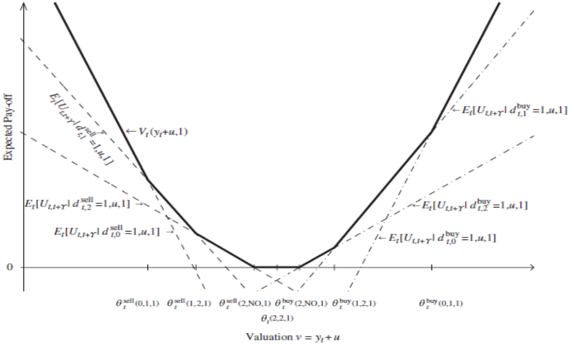

FIGURE 2

Indirect utility function.

The graph is an example of the indirect utility function. The order quantity is set equal to one. The horizontal axis is the trader's valuation, and the vertical axis is the expected pay-off from alternative order submissions. Sell orders are plotted with dashed lines (- - -) and buy orders are plotted with dashed–dotted lines (-.-.). The indirect utility function is plotted with the thick solid line (—). The horizontal axis and the vertical axis have different scale

- Consider a market where:
  - The tick size is one unit: $p_{kt}^{(b)} - p_{0t}^{(b)} = k$.
  - The market buy price is one hundred: $p_{0t}^{(b)} = 100$.
  - There is no common component: $y_t \equiv 0$.
  - Traders submit orders at $p_{0t}^{(b)}$, $p_{1t}^{(b)}$ and $p_{2t}^{(b)}$.
  - By definition $\psi_{0t}^{(b)} = 1$.
  - Also assume $\psi_{1t}^{(b)} = 0.7$ and $\psi_{2t}^{(b)} = 0.6$.
- Using the formula for calculating threshold valuations:

$$\begin{aligned} \theta_t^{(b)}(0,1) &= 100 + 0.7 \big/ (1 - 0.7) = 102.33 \\ \theta_t^{(b)}(1,2) &= 99 + 0.6 \big/ (0.7 - 0.6) = 105.00 \end{aligned}$$

- Since $\theta_t^{(b)}(1,2) > \theta_t^{(b)}(0,1)$ the monotonicity condition is violated.
- In Figure 4 (next slide) the solid line for indirect utility lies strictly above the utility benefit from submitting a limit buy order at 99.

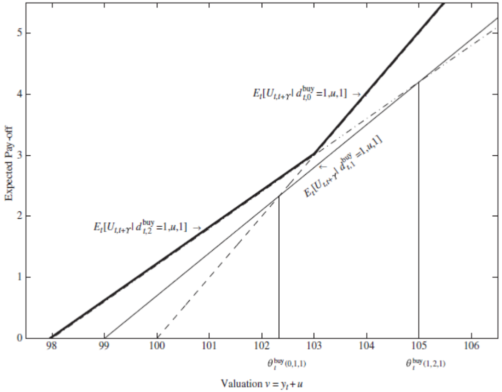

FIGURE 4

Example of violation of the monotonicity of the thresholds.

The figure is an example where the threshold valuations do not satisfy the monotonicity condition $\theta^{buy}_t(0, 1, 1) < \theta^{buy}_t(1, 2, 1)$. The execution probabilities for limit orders are monotonically decreasing in the distance between the limit order price and the best ask quote. The execution probabilities are $\psi^{buy}_t(0, 1) = 1$, $\psi^{buy}_t(1, 1) = 0{\cdot}7$, $\psi^{buy}_t(2, 1) = 0{\cdot}6$; the tick size is 1; the best ask quote is 100; and the picking off risks are equal to zero. The expected pay-off for a trader submitting a buy market order (- - -), a one tick buy limit order (—) and a two tick buy limit order (-.-.) are plotted as a function of the trader's valuation

- Given a typical inequality implied by the model, say:

$$\theta_t^{(b)}(k, k+1) - \theta_t^{(b)}(k+1, k+2) > 0$$

for any $z_t \in \mathcal{F}_t$, where $\mathcal{F}_t$ denotes the information set of trader t:

$$E\left[\theta_t^{(b)}(k, k+1) - \theta_t^{(b)}(k+1, k+2)\,|z_t\right] > 0$$

which implies:

$$E\left\{\left[\theta_t^{(b)}(k, k+1) - \theta_t^{(b)}(k+1, k+2)\right]|z_t|\right\} > 0$$

- The test statistic is based on:

$$T^{-1}\sum_{t=1}^{T}\left\{\left[\widetilde{\theta}_t^{(b)}(k, k+1) - \widetilde{\theta}_t^{(b)}(k+1, k+2) - LB\right]|z_t|\right\}$$

where:

  - $\widetilde{\theta}_t^{(b)}(k, k+1)$ is a consistent estimator for $\theta_t^{(b)}(k, k+1)$
  - and $0 < LB \leq \theta_t^{(b)}(k, k+1) - \theta_t^{(b)}(k+1, k+2)$ for all (t, k, z_t).

Testing an LOM model

Overview of test procedure

- To implement the test we must first identify the
  1. find subsequences of conditional choice submission probabilities $\left\{\lambda_{jt}^{(b)}\right\}_j$ and $\left\{\lambda_{jt}^{(s)}\right\}_j$ that are strictly positive
  2. estimate the execution probabilities $\psi_{kt}^{(b)}$ and $\psi_{kt}^{(s)}$ for the elements in the subsequence
  3. estimate the y_t process
  4. estimate the picking-off risk $\xi_{kt}^{(b)}$ and $\xi_{kt}^{(s)}$
  5. form the threshold values $\theta_t^{(b)}(k, k')$ and $\theta_t^{(s)}(k, k')$.
  6. test the inequalities that apply to $\theta_t^{(b)}(k, k')$ and $\theta_t^{(s)}(k, k')$.
- The null hypothesis is a joint hypothesis combining all six steps.
- Therefore the *size* of the test (the probability of being in the tail of a test statistic) is affected by all the sources of sampling variation.

Testing an LOM model

Notes on implementation in HMS (2004)

- The sample comprises data on Ericsson taken from the Stockholm Automated Exchange system in 1991- 92.
- We focus on $\left\{ p_{0t}^{(b)}, p_{1t}^{(b)}, p_{2t}^{(b)}, p_{3t}^{(b)} \right\}$ and $\left\{ p_{0t}^{(s)}, p_{1t}^{(s)}, p_{2t}^{(s)}, p_{3t}^{(b)} \right\}$.
- We conducted the tests of strictly positive submission probabilities, strictly positive differences in execution probabilities, and monotonicity in threshold valuations separately.
- Consequently the critical values for the tests aren't adjusted properly for sampling error in prior stages.
- We cannot reject the (separately tested) hypotheses that for probabilities of:

$$\begin{array}{l} \text{submission } \lambda_{jt}^{(b)} > 0 \text{ and } \lambda_{kt}^{(b)} > 0 \text{ for } j \in \{0, 1, 2, 3\} \\ \text{execution } \psi_{jt}^{(b)} > \psi_{j+1,t}^{(b)} \text{ and } \psi_{jt}^{(s)} > \psi_{j+1,t}^{(s)} \text{ for } j \in \{0, 1, 2\} \end{array}$$

Test Results

Monotonicity test results (HMS 2004, Table 8, page 1052)

TABLE 8

Monotonicity tests for the threshold valuations

Threshold valuation difference	Constant	Order quantity	Ask depth	Bid depth	Lagged volume	Index volatility	Time of day	Joint M_{PC} statistic
				Instruments				
				Buy threshold valuations				
$\theta^{buy}(0, 1, X_t)$	2·15	14·82	4·66	5·11	5·38	1·42	25·76	0·00
$-\theta^{buy}(1, 2, X_t)$	(0·15)	(1·03)	(0·37)	(0·40)	(0·40)	(0·12)	(1·84)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
$\theta^{buy}(1, 2, X_t)$	1·21	8·34	2·73	3·02	2·93	0·78	14·58	0·00
$-\theta^{buy}(2, 3, X_t)$	(0·14)	(0·93)	(0·33)	(0·35)	(0·38)	(0·12)	(1·68)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
				Sell threshold valuations				
$\theta^{sell}(1, 2, X_t)$	2·02	13·89	4·58	5·06	5·02	1·32	24·28	0·00
$-\theta^{sell}(0, 1, X_t)$	(0·16)	(1·20)	(0·38)	(0·44)	(0·44)	(0·14)	(1·83)	
	1·00	1·00	1·00	1·00	1·00	1·00	1·00	0·99
$\theta^{sell}(2, 3, X_t)$	0·24	1·61	0·47	0·54	0·57	0·16	2·82	0·00
$-\theta^{sell}(1, 2, X_t)$	(0·49)	(3·61)	(1·14)	(1·29)	(1·44)	(0·41)	(5·48)	
	0·69	0·67	0·66	0·66	0·65	0·65	0·70	0·99
				Buy and sell threshold valuations				
$\theta^{buy}(2, 3, X_t)$	−1·43	−9·84	−3·26	−3·62	−3·47	−0·92	−17·27	13·16
$-\theta^{sell}(2, 3, X_t)$	(0·42)	(3·01)	(1·00)	(1·11)	(1·25)	(0·38)	(4·77)	
	0·00	0·00	0·00	0·00	0·00	0·01	0·00	0·01
				Joint $M_{D\theta}$ statistic				
Buy thresholds	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·75	0·76	0·74	0·75	0·75	0·76	0·75	1·00
Sell thresholds	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	0·75	0·75	0·75	0·75	0·75	0·75	0·75	1·00
Buy and sell thresholds	70·35	76·33	80·97	76·43	50·46	32·90	79·49	99·31
	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00

The top three panels of the table report the average differences of threshold valuations for different order prices multiplied by positive instruments. Asymptotic standard errors in parentheses and the p-values are reported below the point estimates. The rightmost column and the bottom panel of the table report joint $M_{D\theta}$ test statistics across the instruments, the order prices, and across instruments and order prices, with p-values reported below each test statistic. We ensure that all instruments are strictly positive by replacing them with 0·00001 if they are zero.

Empirical Content of LOM models

Where does the model succeed?

- For several components of $|z_t|$ the estimated differences:

$$E\left\{\left[\theta_t^{(b)}(j,j+1)-\theta_t^{(b)}(j+1,j+2)\right]|z_t|\right\}$$

  are positive and significant, as the model predicts.
- On the sell side:

$$E\left\{\left[\theta_t^{(s)}(1,2)-\theta_t^{(s)}(0,1)\right]|z_t|\right\}$$

  is positive and significant, but:

$$E\left\{\left[\theta_t^{(s)}(2,3)-\theta_t^{(s)}(1,2)\right]|z_t|\right\}$$

  is positive but not significant.
- Summarizing, the null hypothesis of monotonicity is not rejected when buy and sell thresholds are considered separately.

Test Results

Where does the model fail?

- Contrary to the predictions of the model, the sample:
  - gives negative point estimates of:

$$E\left\{\left[\theta_t^{(b)}(2,3) - \theta_t^{(s)}(2,3)\right]|z_t|\right\}$$

  - rejects the null hypothesis that buyer threshold valuations are higher than seller threshold valuations.
- Thus rejections only occur for investors who are almost indifferent between placing a high limit sell order versus a low limit buy order.
- According to our parameter estimates, as the next slide illustrates:
  1. investors placing high sell limit orders should be placing low buy limit orders instead.
  2. investors placing low buy limit orders should be placing high limit sell orders instead.

Test Results

Illustrating the model rejection (Figure 5, HMS,2004)

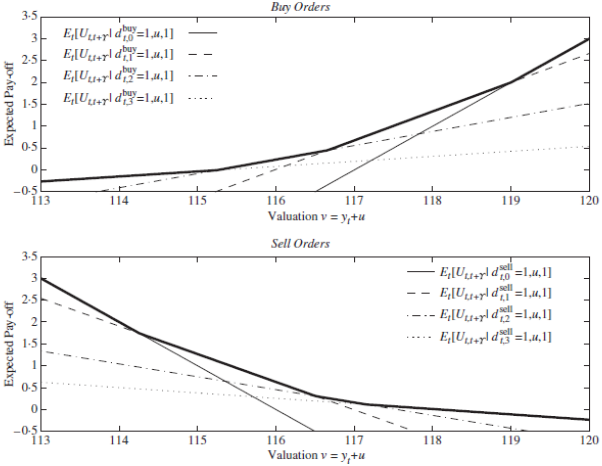

FIGURE 5

Estimated pay-offs.

The figure plots the estimated pay-offs as a function of the trader's valuation. The estimated pay-offs are evaluated at the sample observation with conditioning variables closest to their sample averages. The horizontal axis gives the trader's valuation and the vertical axis the pay-offs for alternative order submissions

- We now assume traders arrive sequentially at rate:
  - $\Pr\{\text{Trader arrives in interval } [t, t + \Delta t] \, | x_t \} = \lambda \, (t; x_t) \, dt$
  - where x_t is an exogenous vector of state variables
- Following the same notation as before:
  - $d_{kt}^{(b)} \in \{0, 1\}$ and $d_{kt}^{(s)} \in \{0, 1\}$ for $k \in \{0, 1, 2, \ldots\}$
  - with the same constraint $\sum_{k=0}^{\infty} \left(d_{kt}^{(b)} + d_{kt}^{(s)} \right) \leq 1.$
- As above we assume traders are risk neutral with valuations:
  - differ in their private valuation $v_t = y_t + u_t$
  - where u_t is distributed independently with $\Pr\left(u_t \leq u \, | x_t \right) \equiv G\left(u \, | x_t \right)$
- If and when he has the opportunity the trader:
  - can submit an order to trade one unit
  - pays c_0 to placing an order
  - pays a further c_e if the order executes.

- Note that $G(u|x)$ must be estimated to obtain estimates of the gains from trade $L(z_t)$.
- One strategy is to:
  1. Follow the same procedure as above to:
     - determine orders with positive submission probabilities $\lambda_{jtj}^{(b)}$ and $\lambda_{jt}^{(s)}$.
  2. Then estimate:
     - their execution probabilities $\psi_{kt}^{(b)}$ and $\psi_{kt}^{(s)}$ (nonparametrically).
     - the y_t process.
     - the picking-off risks $\xi_{kt}^{(b)}$ and $\xi_{kt}^{(s)}$.
  3. Apply a competing risk hazard framework to jointly estimate:
     - the arrival rate of traders $\lambda(t; x_t)$.
     - and $G(u|x)$, the distribution of their valuations.

- Briefly, we partition each time interval $[t, t + dt)$ by each possible event, and estimate the probability of its occurrence:
- For example a crude partition of events is that in $[t, t + dt)$:

1. A market buy order arrives:

$$\Pr\left\{\widehat{d}_{0t}^{(b)} = 1 \text{ in } [t, t+dt)\,|z_t\right\} = \left\{1 - G\left[\theta_t^{(b)}(0,1)\,|x\right]\right\}\lambda(t; x_t)\,dt$$

2. There is a market sell order:

$$\Pr\left\{\widehat{d}_{0t}^{(s)} = 1 \text{ in } [t, t+dt)\,|z_t\right\} = G\left[\theta_t^{(s)}(0,1)\,|x\right]\lambda(t; x_t)\,dt$$

3. Either a limit order arrives or there is no order:

$$\begin{aligned}&\Pr\left\{\widehat{d}_{0t}^{(b)} + \widehat{d}_{0t}^{(s)} = 0 \text{ in } [t, t+dt)\,|z_t\right\}\\ = \;& 1 - \lambda(t; x_t)\,dt\\ &+ \left\{G\left[\theta_t^{(b)}(0,1)\,|x\right] - G\left[\theta_t^{(b)}(0,1)\,|x\right]\right\}\lambda(t; x_t)\,dt\end{aligned}$$

How Efficient is an LOM?

Equilibrium gains from trade from behind the Rawlsian (1971) veil of ignorance

- The gains from trade do not depend on the transaction price or the picking off risk, which are transfers between buyer and seller.
- When the buyer places a limit order at t and the seller places a market order at $t+\tau$ cancelling the buy order, the gains from trade are:

$$u_{t+\tau} - u_t - 2\left(c_0 + c_e\right)$$

- More generally, the expected gains from a new trader arriving at t are:

$$V = E\left\{ \begin{array}{c} \sum_{k=0}^{\infty} \widehat{d}_{kt}^{(b)}\left(u_t, z_t\right)\left[\psi_{kt}^{(b)}\left(z_t\right)\left(u_t - c_e\right) - c_0\right] \\ -\sum_{k=0}^{\infty} \widehat{d}_{kt}^{(s)}\left(u_t, z_t\right)\left[\psi_{kt}^{(s)}\left(z_t\right)\left(c_e + u_t\right) + c_0\right] \end{array} \right\}$$

How Efficient is an LOM?

Maximal gains from exchange from behind the Rawlsian (1971) veil of ignorance

- We compare the expected gains from trade in an LOM with the *potential gains from exchange*, obtained by choosing between:
  - immediately executing a new order
  - or placing the order in inventory
  - where orders in the inventory are subjected to cancellation risk
  - to maximize the expected gains from exchange.
- We can categorize the reasons why limit order markets do not realize all the potential gains from exchange.
  1. Limit orders are not executed when they should be.
  2. Traders do not submit orders when they should.
  3. Trader submits a "wrong sided" order that executes.
  4. Traders submit orders when they should not.
- Our estimates understate the efficiency of LOMs, because the maximal gains from trade we compute do not account for coordination costs investors face by arriving at the market at different times.

The Vancouver Stock Exchange (VSE)

Structural estimates from Hollifield, Miller, Sandas and Slive (2006)

	BHO	ERR	WEM
	Gains		
	Maximum gains as a % of the common value		
	9.07	8.61	6.75
	Current gains as a % of the common value		
Lower bound	7.88	8.09	6.08
Upper bound	8.45	8.31	6.40
Average	8.16	8.20	6.24
	Maximum gains minus current gains		
Lower bound	0.62	0.30	0.35
Upper bound	1.20	0.52	0.67
Average	0.91	0.41	0.51
	Current gains as a % of maximum gains		
Lower bound	86.79	93.97	90.07
Upper bound	93.13	96.57	94.81
Average	89.96	95.27	92.44
	Decomposition of Losses		
	No execution as a % of total losses		
Sell side	32.32	31.20	33.05
Buy side	40.10	39.01	41.85
Subtotal	72.42	70.21	74.90
	No submission as a % of total losses		
Sell side	2.24	0.62	0.41
Buy side	1.98	0.15	0.71
Subtotal	4.22	0.77	1.12
	Wrong direction as a % of total losses		
Sell side	0.86	0.02	0.39
Buy side	0.20	0.05	0.63
Subtotal	1.06	0.07	1.02
	Extramarginal submissions as a % of total losses		
Sell side	9.81	11.87	10.30
Buy side	12.49	17.07	12.66
Subtotal	22.30	28.94	22.96
Total	100.00	100.00	100.00
	Monopoly Gains		
	Monopoly gains as a % of the common value		
	5.02	5.57	4.18
	Monopoly gains as a % of maximum gains		
	55.31	64.71	61.87
	Current gains as a % of monopoly gains		
	162.65	147.23	149.41

The Vancouver Stock Exchange

Received history

- A brief history of the VSE:
  - Incorporated 1906, and fully automated in 1990.
  - Trading increased from C$4 billion in 1991 to $6.7 billion in 1993.
  - Merged with Canadian Venture Exchange (CDNX) in 1999.
  - Subsequently absorbed into the Toronto Stock Exchange (TSE).
- VSE had an unsavory reputation reminiscent of the wild west:
  - In 1989, Forbes magazine christened it "scam capital of the world".
  - A 1994 report refers to "shams, swindles and market manipulations".
  - The summary judgement of Investopedia.com is that "the VSE is an example of one of the world's less successful stock exchanges."
- The historical narrative of VSE is puzzling:
  1. Our analysis paints a glowing picture of capitalism at work.
  2. Why did the trading volume grow substantially after automation?
  3. Several European exchanges merged when the VSE was absorbed:
     - Is this evidence of unsuccessful exchanges?
     - Was this driven by the electronic exchange technology?