CEMMAP Lectures on Contracts & Market Microstructure

2. Sealed Bid Auctions

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Introduction

How useful are models of competitive equilibrium in assessing conduct and performance?

- The two welfare theorems nicely encapsulate the close connection between exploiting the potential gains from trade and a price vector separating suppliers from demanders.
- But how is price set in competitive equilibrium?
  1. We imagine a fictional *Walrasian auctioneer*
  2. setting *market clearing* prices
  3. by maximizing *a weighted sum of individual utilities*
  4. to mimic a *centralized* omniscient social planner.
- Competitive equilibria are not *implementable*:
  - loosely speaking a noncooperative game with a Nash equilibrium that reproduces the competitive allocations does not exist.
- Furthermore the empirical results inferred from structural econometric models of competitive equilibrium, about how well the private economy performs, are ambiguous.

- This is a motivation for exploring the institutional structure of markets, digging more deeply into how markets function.
- An obvious place to start these investigation are auction mechanisms:
  1. There is single agent on one side of the market, the auctioneer.
  2. In the simplest case only one (indivisible) is traded.
  3. Can apply standard noncooperative definitions of games
     - players, moves, information sets and payoffs
     - Nash equilibrium refinements (such as perfection)
- (Variations on) these models can show how:
  1. the number of traders on the other side of the market determine the equilibrium.
  2. endogenous search for a market affects the trading outcome.
  3. what factors affect the choice of the trading mechanism.

- **Strategically equivalent** auctions:
  - first price sealed bid (FPSB) auctions
    (*highest bidder wins and pays his bid*).
  - Dutch auctions
    (auctioneer reduces the price until a bidder accepts the offer)
- Strategically equivalent auctions when there are **private values**:
  - second price sealed bid (SPSB) auctions
    (highest bidder wins, paying highest losing bid).
  - Japanese (button) auctions players exit as the auctioneer raises the price
    (*last remaining bidder wins at price second last exits*).

First Price Sealed Bid (FPSB) Auction

Basic framework

- We first consider a first price sealed bid (FPSB) auction:
  - for N players (predetermined outside the model)
  - with independent private values.
- By FPSB we mean that each player $n \in \{1, \ldots, N\}$ simultaneously submits a bid denoted by $b_n \in \mathbf{R}^+$, and that the player submitting the highest bid is awarded the (single) object up for auction, and pays what he or she bid.

First Price Sealed Bid (FPSB) Auction

Best replies in equilibrium

- Let $W(b)$ denote the probability of winning the auction with bid b. That is:

$$W(b) \equiv \Pr\{b_k \leq b \text{ for all } k = 1, \ldots, N\}$$

- Then the maximization problem faced by player n can be written as:

$$\max_b (v_n - b) W(b)$$

- The first order condition (FOC) is:

$$(v_n - b_n) W'(b_n) - W(b_n) = 0 \tag{1}$$

- The second order condition (SOC) of the optimization problem is:

$$\begin{aligned} 0 > SOC \equiv \frac{\partial}{\partial b} FOC &= \frac{\partial}{\partial b}\left[(v - b) W'(b) - W(b)\right] \\ &= (v - b) W''(b) - 2W'(b) \end{aligned}$$

- Totally differentiating the FOC with respect to b and v yields:

$$0 = W'(b_n)\, dv_n + \left[(v_n - b_n) W''(b_n) - 2W'(b_n)\right] db_n$$

and hence:

$$\frac{db_n}{dv_n} = \frac{-W'(b_n)}{(v_n - b_n) W''(b_n) - 2W'(b_n)} > 0$$

because $W'(b_n) > 0$ and the denominator of the quotient is the SOC.

- We infer that if players are in a pure strategy equilibrium with an interior solution, then b_n is increasing in v_n.

First Price Sealed Bid (FPSB) Auction

Bayesian Nash Equilibrium with monotone bidding

- From now on we assume that players are in a (pure strategy) Bayesian equilibrium with bids that are monotone increasing in valuations.
- That is we consider Bayesian Nash Equilibrium (BNE) in which bidders follow a strategy $\beta : \mathbf{V} \rightarrow \mathbf{B} \equiv [0, \infty)$ where $\beta(v)$ is increasing in v.
- Then $\beta(v)$ has an inverse, which we denote by $\alpha : \mathbf{B} \rightarrow \mathbf{V}$ such that $\alpha[\beta(v)] = v$ for all v.
- Letting $G(b)$ denote the distribution of bids, it follows that:

$$W(b) \equiv \Pr\{b_k \leq b_n \text{ for all } k = 1, \ldots, N\} = G(b_n)^{N-1}$$

- From the monotonicity property of the BNE:

$$G(b) = F(\alpha(b))$$

Identifying IPV from FPSB auctions

From the probability of winning

- Assume our data set consists of all the bids recorded in I auctions in which the same equilibrium is played.
- Let b_n^i for $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, I\}$ denote the bid by player n in the i^{th} auction.
- Suppose the probability of winning the i^{th} auction, $W_i(b)$, and hence its derivative $W_i'(b)$, are identified.
- Rewriting the FOC (1) then proves v_n^i is identified:

$$v_n^i = b_n^i + \frac{W_i\left(b_n^i\right)}{W_i'\left(b_n^i\right)} \tag{2}$$

- For example assume $v_n \in V$ is iid (IPV) drawn from $F(v)$, then $W(b)$, v_n^i and $F(v)$ are identified.
- This logic extends to models where v_n is drawn from $F(v | z_n)$ where:
  - each bidder knows z_n, some background variables.
  - bidders know, or do not know, the z values of their rivals.
  - The analyst observes z_n and bidders' information sets.

Identifying IPV from FPSB auctions

From the bid distribution (Guerre, Perrigne and Voung, 2003)

- Alternatively note that the probability distribution of bids and its density, $G_i(b)$ and $G_i'(b)$, are identified.
- Focusing on the IPV case (purely for notational ease) the probability n wins with b_n is:

$$W(b_n) = G(b_n)^{N-1}$$

implying

$$W'(b_n) = (N-1)\,G(b_n)^{N-2}G'(b_n)$$

- We rewrite the FOC, Equation (1) as:

$$v_n^i = b_n^i + \frac{W(b_n^i)}{W'(b_n^i)} = b_n^i + \frac{G(b_n^i)}{(N-1)\,G'(b_n^i)} \tag{3}$$

- This shows v_n^i and hence $F(v)$ can also be directly identified off the bidding distribution $G(b)$.

- Now suppose our data set consists of only the winning bid recorded in I auctions in which the same equilibrium is played.
- Let b^i for $i \in \{1, \ldots, I\}$ denote the winning bid in the i^{th} auction.
- Thus the distribution of winning bids, denoted by $H\left(b^i\right)$, is identified.
- Since the winning bid is defined as the highest one, $H\left(b\right)$ is just the probability that all the bids are less than b, implying:

$$H\left(b\right) = \Pr\left\{b_n^i \leq b \text{ for all } n = 1, \ldots, N\right\} = G(b)^N$$

- Consequently:

$$G(b) = H\left(b\right)^{\frac{1}{N}} \tag{4}$$

and

$$G'(b) = \frac{1}{N} H\left(b\right)^{\frac{1}{N}-1} H'\left(b\right) \tag{5}$$

- This shows the bidding distribution is identified from the data generating process of the winner's bid.

- Substituting Equations (4) and (5) back into Equation (3) gives:

$$v^i = b^i + \frac{G\left(b^i\right)}{(N-1)\,G'\left(b^i\right)} = b^i + \frac{NH\left(b\right)}{(N-1)\,H'\left(b\right)}$$

- This identifies the winning valuations, and hence their distribution, denoted by $F_W(v)$.
- But the distribution of the winning valuations is a one to one mapping of the distribution of all the valuations:

$$F_W(v) = \Pr\left\{v_n \leq v \text{ for all } n = 1, \ldots, N\right\} = F(v)^N$$

- Therefore $F(v)$ is identified off the winning bids alone using the equation:

$$F(v) = F_W(v)^{\frac{1}{N}}$$

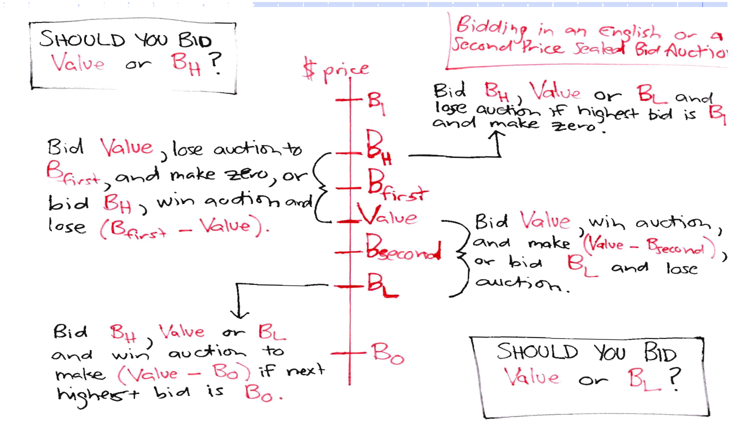
Second Price Sealed Bid (SPSB) Auctions

A weakly dominant strategy

- Now suppose as before:
  - each bidder knows her own valuation;
  - makes sealed bid (that is bids simultaneously).
- But instead of a FPSB auction, consider a SPSB auction, where the highest bidder wins the auction but only pays the second highest bid.
- Now it is a weakly dominant strategy for (each) n to bid her expected valuation, v_n.
- Intuitively, compared with bidding v_n:
  - bidding more implies winning some auctions that yield negative expected value, but leaves unchanged the expected value of any other auction that would be won;
  - bidding less implies losing some auctions that yield positive expected value, but leaves unchanged the expected value of any other auction that she would win.

Second Price Sealed Bid (SPSB) Auctions

A picture proof



Identifying the IPV Distribution from SPSB Auctions

Distribution of the second highest valuation

- Now assume $F(v)$ is the common distribution of valuations as before.
- Note first the obvious point that because players bid their valuations in SPSB auctions with private valuations, $F(v)$ is trivially identified if all the bids are observed.
- Now suppose only the winning price is observed.
- Then the probability distribution of the second highest valuation, which we now denote by $F_{N-1,N}(v)$, is identified.

- More generally, let $F_{i,N}(v)$ denote the distribution of the i^{th} order statistic.
- One can show (Arnold, Balakrishnan and Nagaraja,1992 for example):

$$F_{i,N}(v) = \frac{N!}{(N-i)!\,(i-1)!} \int_0^{F(v)} t^{i-1} (1-t)^{N-i}\, dt \qquad (6)$$

- We can interpret (6) as a mapping from distribution functions in valuations into distribution functions of ordered valuations.
- To prove $F(v)$ is identified, fix v_0 and note:
    1. $F_{i,N}(v_0)$ is identified (and a consistent estimator is obtained from the sample distribution).
    2. The right side of (6) is strictly increasing in candidate values of $F(v_0)$ since $t^{i-1}(1-t)^{N-i} > 0$ for all $t \in [0,1]$.
    3. Therefore the LHS and RHS of (6) cross only once at v_0.
    4. Hence the mapping defining $F_{i,N}(v)$ from $F(v)$ is invertible.

Bidder Information

Revisiting best replies in SPSB auctions with private values

- Suppose there are N risk neutral bidders. Bidder n:
  - has valuation v_n, the utility gain from winning the auction.
  - receives signal $x_n \equiv v_n + \epsilon_n$, where $E[\epsilon_n | x_n] = 0$.
- The literature focuses on perfect Bayesian equilibria in weakly undominated pure strategies (Athey and Haile, 2006).
- Let $b_n \equiv \beta_n(x_n, N)$ denote the equilibrium strategy of bidder n.
- In a second price auction with private values, it is a weakly dominant strategy for (each) n to bid his expected valuation, setting:

$$\beta_n(x_n, N) = x_n \equiv E[v_n | x_n]$$

- Note the same logic applies to n individually if $v_n = x_n$, regardless of the correlation structure between valuations and the other bidders' information.

Bidder Information

Revisiting best replies in FPSB auctions with private values

- Denote the bid distribution function for the maximum equilibrium bid of the n^{th} bidder's rivals, which is independent of x_n, as:

$$G\left(b_m \left| n, N\right.\right) = \Pr\left[\max_{n' \in N \backslash n} \left\{b_{n'}\right\} \leq b_m \left| n, N\right.\right]$$

- Then b_n solves:

$$b_n = \arg\max_b \int_{-\infty}^{b} \left(x_n - b\right) G'\left(b_m \left| n, N\right.\right) db_m$$

- The first order condition is:

$$x_n = b_n + \frac{G\left(b_n \left| n, N\right.\right)}{G'\left(b_n \left| n, N\right.\right)}$$

- This FOC reduces to (2) when $v_n = x_n$ and the valuations of the bidders are *iid.*
- In both cases both $W\left(b_n\right)$ and $G\left(b_n \left| x_n, N\right.\right)$ represent the probability of n winning the auction with bid b_n.

- Similar to the private values case, define:

$$\widetilde{G}\left(b_m \left| x_n, n, N\right.\right) = \Pr\left[\max_{n' \in N \backslash n}\left\{b_{n'}\right\} \leq b_m \left| x_n, n, N\right.\right]$$

- Suppose the equilibrium is symmetric, and
    - denote by $\beta\left(x_{n'}, N\right)$ denote of the bid function of n'.
    - suppose $\beta\left(x_{n'}, N\right)$ is strictly increasing in $x_{n'}$.
    - denote by $\alpha\left(b_{n'}, N\right)$ the inverse of $\beta\left(x_{n'}, N\right)$ in $x_{n'}$.
- Then the optimal bid b_n solves:

$$b_n = \arg\max_b \int_{-\infty}^{b}\left[v_n\left(x_n, \alpha\left(b_m, N\right), N\right) - b\right]\widetilde{G}'\left(b_m \left| x_n, n, N\right.\right) db_m$$

where $v_n\left(x_n, x_m, N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \backslash n}\left\{b_{n'}\right\} = \beta\left(x_m, N\right)\right]$

- Superficially, this FOC resembles the private value auction FOC:

$$v_n\left(x_n, x_n, N\right) = b_n + \widetilde{G}\left(b_n \left| x_n, n, N\right.\right) \Big/ \widetilde{G}'\left(b_n \left| x_n, n, N\right.\right).$$

- Assume $x_n = v_n$. From the first order condition:

$$x_n = b_n + \frac{G\left(b_n \left| n, N\right.\right)}{G'\left(b_n \left| n, N\right.\right)}$$

- Recall from its definition that $G_{m_n}\left(b_n \left| n, N\right.\right)$ is the probability that n wins the auction with b_n:

$$G_m\left(b_n \left| n, N\right.\right) = \Pr\left[\max_{n' \in N \backslash n}\left\{b_{n'}\right\} \leq b_n \left| n, N\right.\right]$$

- Thus if all the bids are observed then $G_m\left(b_n \left| x_n, N\right.\right)$ is identified.
- Hence v_n is identified (for all bidders in each sampled auction).
- Therefore the probability distribution of $\left(v_1, \ldots, v_N\right)$ in this specialization is identified for any correlation structure.

Identification Fails in Common Value FPSB Auctions

When all the bids are observed

- Recall that we defined:

$$v_n\left(x_n, x_m, N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \backslash n}\left\{b_{n'}\right\} = \beta_n\left(x_m, N\right)\right]$$

and derived:

$$v_n\left(x_n, x_n, N\right) = b_n + \frac{\widetilde{G}\left(b_n \left| x_n, n, N\right.\right)}{\widetilde{G}'\left(b_n \left| x_n, n, N\right.\right)}$$

- The basic problem is that the function $v_n\left(x_n, x_m, N\right)$ is only identified at the point $x_n = x_m$, not any x_m.
- Note that every common value model is observationally equivalent to a private value model found by setting $v_n = v_n\left(x_n, x_n', N\right)$.
- Thus two common value models with possibly different $v_n\left(x_n, x_m, N\right)$ but the same $v_n\left(x_n, x_n, N\right)$ are (also) observationally equivalent.