Finite Dependence

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Defining Finite Dependence

Summaring our results on identification from last lecture

- The last lecture demonstrated that if (T, β, f, g) is known, then without further restrictions:
  1. u is not identified in long panels unless one choice specific payoff for each state is known;
  2. even if one choice specific payoff for each state is known u is not identified in short panels;
  3. counterfactuals for payoff innovations are identified off short panels;
  4. counterfactuals for transition innovations are not identified off long panels unless one choice specific payoff for each state is known;
  5. counterfactuals for transition innovations are not identified off short panels.

Defining Finite Dependence

Surveying the way forward

- This summary suggests two directions for exploration:
  1. The theorem on set identification also provides some guidance about what kinds of additional restrictions are necessary to relax the assumption that one choice specific payoff for each state is known, and indeed that (T, β, f, g) is known. The additional restrictions must shrink the u^* set, by ruling out payoff flows that would otherwise be observationally equivalent.
  2. Placing restrictions on u might potentially identify counterfactuals for transition innovations from short panels.
- In this fourth lecture on identification, we show how the combination of finite dependence and stable payoffs addresses both issues.

Defining Finite Dependence for Optimization Problems

Weighted distribution of state variables induced by weighted choices

- Let $\omega_{jkt\tau}(x_t, x_{t+\tau})$ denote the weight on the k^{th} action at period $t+\tau$ when the state is $x_{t+\tau}$, was x_t at t, and action j was taken at t.
- Let:

$$\omega_{jt\tau}(x_t, x_{t+\tau}) \equiv (\omega_{j2t\tau}(x_t, x_{t+\tau}), \ldots, \omega_{jJt\tau}(x_t, x_{t+\tau}))$$

  where $\sum_{k=1}^{J} \omega_{jkt\tau}(x_t, x_{t+\tau}) = 1$.
- We recursively define a weight distribution by setting $\kappa_{jt0}(x_{t+1}|x_t) \equiv f_{jt}(x_{t+1}|x_t)$, and:

$$\kappa_{jt\tau}(x_{t+\tau+1}|x_t) \equiv \sum_{x=1}^{X} \sum_{k=1}^{J} f_{k,t+\tau}(x_{t+\tau+1}|x)\omega_{jkt\tau}(x_t, x)\, \kappa_{jt,\tau-1}(x|x_t)$$

- In the special case where $\omega_{jkt\tau}(x_t, x_{t+\tau}) \geq 0$ for all $(j, k, \tau, x_t, x_{t+\tau})$, then ω can be interpreted as a randomized decision rule, and $\kappa_{jt,\tau-1}(x_{t+\tau}|x_t)$ the probability of reaching $x_{t+\tau}$ in period $t+\tau$ from (t, x_t) by taking choice j at t and then applying ω.

Defining Finite Dependence

Equalizing the weight distribution of state variables for a pair of paths

- Consider two sequences of decision weights beginning at date t in state x_t, one with choice i and the other with choice j.
- We say that the pair of choices (i, j) exhibits ρ-period dependence at (t, x_t) if there exists an ω from i and j for x_t such that for all $x_{t+\rho+1}$:

$$\kappa_{it\rho}(x_{t+\rho+1}|x_t) = \kappa_{jt\rho}(x_{t+\rho+1}|x_t)$$

- That is, the weights associated with each state are the same across the two paths after ρ periods.
- Finite dependence trivially holds in all finite horizon problems, but ρ-period dependence only merits attention when $\rho < T - t$.
- For this reason we ignore the trivial case of $\rho = T - t$.

Defining Finite Dependence in Games

Reduced form transitions and weights in dynamic games

- We specialize by assuming each player n can only directly affect its part of the state space, partitioned as $x_t = \left(x_t^{(1)}, \ldots, x_t^{(N)} \right)$:

$$\Pr \left\{ x_{t+1} \left| x_t, d_t \right. \right\} = \prod_{n=1}^{N} \left[\sum_{k=1}^{J} d_{kt}^{(n)} F_{kt}^{(n)} \left(x_{t+1}^{(n)} \left| x_t^{(n)} \right. \right) \right]$$

- Following Lecture 5 let:

$$f_t^{(\sim n)} \left(x_{t+1}^{(\sim n)} \left| x_t \right. \right) \equiv \prod_{\substack{n'=1 \\ n' \neq n}}^{N} \left[\sum_{k=1}^{J} p_{kt}^{(n')} \left(x_t \right) F_{kt}^{(n')} \left(x_{t+1}^{(n')} \left| x_t^{(n')} \right. \right) \right]$$

- Then the probability of reaching x_{t+1} from x_t when n chooses j and all the other players use their equilibrium strategy is:

$$f_{jt}^{(n)} \left(x_{t+1} \left| x_t \right. \right) \equiv F_{jt}^{(n)} \left(x_{t+1}^{(n)} \left| x_t^{(n)} \right. \right) f_t^{(\sim n)} \left(x_{t+1}^{(\sim n)} \left| x_t \right. \right)$$

Defining Finite Dependence in Games

Notation used to extend framework to dynamic games

- Consider for all $\tau \in \{1, \ldots, T - t\}$ any sequence of decision weights:

$$\omega_{jt\tau}^{(n)}(x_t, x_{t+\tau}) \equiv \left(\omega_{j1t\tau}^{(n)}(x_t, x_{t+\tau}), \ldots, \omega_{jJt\tau}^{(n)}(x_t, x_{t+\tau})\right)$$

  subject to the constraint:

$$\sum_{k=1}^{J} \omega_{jkt\tau}^{(n)}(x_t, x_{t+\tau}) = 1.$$

- We now recursively define a weight distribution by setting $\kappa_{jt0}(x_{t+1}|x_t) \equiv f_{jt}^{(n)}(x_{t+1}|x_t)$, and:

$$\kappa_{jt\tau}^{(n)}(x_{t+\tau+1}|x_t) \equiv \sum_{x=1}^{X} \sum_{k=1}^{J} f_{k\tau}^{(n)}(x_{t+\tau+1}|x)\omega_{jkt\tau}(x_t, x)\, \kappa_{jt,\tau-1}^{(n)}(x|x_t)$$

- ρ-period dependence attains for n at (i, j) given (t, x_t) if there exists a $\omega^{(n)}$ satisfying:

$$\kappa_{it\rho}^{(n)}(x_{t+\rho+1}|x_t) \equiv \kappa_{jt\rho}^{(n)}(x_{t+\rho+1}|x_t)$$

Why Finite Dependence Matters

Representation one more time

- The same telescoping arguments used in proving the representation theorem are useful in showing that:

$$
\begin{aligned}
& v_{jt}(x_t) - u_{jt}(x_t) \\
= \; & \sum_{\tau=t+1}^{t+\rho} \sum_{(k,x_\tau)}^{(J,X)} \beta^{\tau-t} \left\{ \begin{array}{l} \left[u_{k,t+\tau}(x_{t+\tau}) + \psi_{k,t+\tau}(x_{t+\tau})\right] \\ \times \omega_{jkt\tau}(x_t, x_{t+\tau}) \kappa_{jt,\tau-1}(x_\tau | x_t) \end{array} \right\} \\
& + \sum_{x_{t+\rho+1}}^{X} \beta^{t+\rho+1-t} V_{t+\rho+1}(x_{t+\rho+1}) \kappa_{jt\rho}(x_{t+\rho+1} | x_t)
\end{aligned}
$$

Why Finite Dependence Matters

Differencing out the continuation value

- If ρ-period dependence holds at (i, j, t, x) then for some ω:

$$\kappa_{t+\rho}(x_{t+\rho+1}|x, i) = \kappa_{t+\rho}(x_{t+\rho+1}|x, j)$$

- Differencing with respect to i and j, the terms involving $V_{t+\rho+1}(x_{t+\rho+1})$ cancel out leaving:

$$\begin{aligned} & u_{jt}(x) - u_{it}(x) - \psi_{jt}(x) + \psi_{it}(x) \\ = & \sum_{\tau=t+1}^{t+\rho} \sum_{(k,x_\tau)}^{(J,X)} \beta^{\tau-t} \left\{ \begin{array}{c} \left[u_{k,t+\tau}(x_{t+\tau}) + \psi_{k,t+\tau}(x_{t+\tau})\right] \times \\ \left[\begin{array}{l} \omega_{ikt\tau}(x_t, x_{t+\tau}) \kappa_{it,\tau-1}(x_{t+\tau}|x_t) \\ -\omega_{jkt\tau}(x_t, x_{t+\tau}) \kappa_{jt,\tau-1}(x_{t+\tau}|x_t) \end{array} \right] \end{array} \right\} \end{aligned}$$

- Without making any further assumptions, these equations are not useful in identification because $u_{jt}(x_t)$ depends on $u_{k\tau}(x_\tau)$ for $\tau > t$ but $u_{k\tau}(x_\tau)$ depends on $u_{k'\tau'}(x_{\tau'})$ for $\tau' > \tau$ and so on.

Why Finite Dependence Matters

Stable utilities

- Suppose we asume that utilities are stable over time, meaning $u_{jt}(x) = u_j(x)$ for all (j, t, x).
- This simplifies the utility representation on the previous slide to:

$$u_j(x) - u_i(x) - \psi_{jt}(x) + \psi_{it}(x)$$

$$= \sum_{\tau=t+1}^{t+\rho} \sum_{(k,x_\tau)}^{(J,X)} \beta^{\tau-t} \left\{ \begin{array}{c} [u_k(x_\tau) + \psi_{k\tau}(x_\tau)] \times \\ \left[\begin{array}{c} \omega_{ikt\tau}(x_t, x_{t+\tau}) \kappa_{it,\tau-1}(x_{t+\tau}|x_t) \\ -\omega_{jkt\tau}(x_t, x_{t+\tau}) \kappa_{jt,\tau-1}(x_{t+\tau}|x_t) \end{array} \right] \end{array} \right\}$$

- There are $X \times T \times J!$ equations but only $X \times T \times (J-1)$ are linearly independent at most.
- In a stationary model, where $\omega_{ikt\tau}(x_t, x_{t+\tau}) \equiv \omega_{ik\tau}(x_t, x_{t+\tau})$ and $\kappa_{jt,\tau-1}(x_{t+\tau}|x_t) \equiv \kappa_{i,\tau-1}(x_{t+\tau}|x_t)$, the number of linearly independent equations falls to $X \times (J-1)$.
- But in a nonstationary model there are many more equations than unknowns.

Examples of Finite Dependence

Terminal choices and stable utility

- A terminal choice ends the evolution of the state variable with an absorbing state that is independent of the current state.
- That is $f_{1t}(x_{t+1}|x) \equiv f_{1t}(x_{t+1})$ for all (t, x).
- Let the first choice denote a terminal choice. Then:

$$\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) = f_{1,t+1}(x_{t+2})$$

- From the representation theorem:

$$\begin{aligned} & u_1(x_t) - u_j(x_t) - \psi_{1t}(x) + \psi_{jt}(x) \\ = & \sum_{x_{t+1}=1}^{X} \beta \left[u_1(x) + \psi_{1,t+1}(x) \right] f_{jt}(x|x_t) \end{aligned}$$

- If there is more than one period of data, and $f_{jt}(x|x_t)$ varies with t, then $u_j(x_t)$ is typically (over) identified for all $j \in \{1, \ldots, J\}$.

Examples of Finite Dependence

Renewal choices and stable utility

- Similarly a renewal choice yields a probability distribution of the state variable next period that does not depend on the current state.
- Letting the first choice denote a renewal choice:

$$\begin{aligned}\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &\equiv \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{1t}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2})\end{aligned}$$

- Normalizing utility for the renewal choice to zero, we obtain from the representation theorem:

$$\begin{aligned}& u_j(x_t) - u_1(x_t) - \psi_{jt}(x) + \psi_{1t}(x) \\ = & \sum_{x=1}^{X} \beta \left[u_1(x) + \psi_{1,t+1}(x) \right] \left[f_{1t}(x|x_t) - f_{jt}(x|x_t) \right]\end{aligned}$$

- The same argument for identification applies.

- How does finite dependence work when $\rho > 1$?
- Consider the following model of labor supply and human capital.
- In each of T periods an individual chooses whether to work, $d_{2t} = 1$, or stay home $d_{1t} = 1$. She acquires human capital, x_t, by working, with the payoff to working increasing in her human capital.
- If the individual works in period t, $x_{t+1} = x_t + 2$ with probability 0.5 and $x_{t+1} = x_t + 1$ also with probability 0.5.
- Every period after t, the human capital gain from working is fixed at one additional unit.
- When the individual does not work, her human capital remains the same in the next period.

Examples of Finite Dependence

Establishing finite dependence in the labor supply example

- First consider staying home at t and then work for the next two periods. Set:

$$\omega_{12t,t+1}(x_t, x_{t+1}) = \omega_{12t,t+2}(x_t, x_{t+2}) = 1$$

- This sequence of choices (stay home, work, work) increases human capital two units by $t+3$.
- Now consider working at t, staying home in period $t+2$, and depending on whether human capital increases by one or two units in t, work in $t+1$. Set:

$$\begin{aligned} \omega_{21t,t+1}(x_t, x_t+2) &= \omega_{21t,t+2}(x_t, x_t+2) = 1 \\ \omega_{22t,t+1}(x_t, x_t+1) &= \omega_{12t,t+2}(x_t, x_t+2) = 1 \end{aligned}$$

- These weights also increase the total the human capital stock by two units for sure.

Examples of Finite Dependence

An alternative way of establishing finite dependence in the labor supply example

- Consider working in period t and then staying home for the next two periods regardless of how much human capital is accumulated:

$$\begin{aligned}\omega_{21t,t+1}(x_t, x_t+2) &= \omega_{21t,t+2}(x_t, x_t+2) = 1 \\ \omega_{21t,t+1}(x_t, x_t+1) &= \omega_{21t,t+2}(x_t, x_t+1) = 1\end{aligned}$$

- Now consider staying home in t, working in $t+1$, and with probability one half working in period $t+2$:

$$\begin{aligned}\omega_{12,t+1}(x_t, x_t) &= 1 \\ \omega_{12,t+2}(x_t, x_t+1) &= \omega_{12t,t+2}(x_t, x_t+1) = 1/2\end{aligned}$$

- In both cases the exante distribution of human capital is the same:

$$\kappa_{1t,t+2}(x_{t+3}|x_t) = \kappa_{2t,t+2}(x_{t+3}|x_t) = \begin{cases} 1/2 & \text{if } x_{t+3} = x_t + 1 \\ 1/2 & \text{if } x_{t+3} = x_t + 2 \end{cases}$$

Examples of Finite Dependence

Nonstationary search model

- Consider a simple search model in which all jobs are temporary, last only one period.
- Each period $t \in \{1, \ldots, T\}$ an individual may stay home by setting $d_{1t} = 1$, or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t
- The current utility from employment depends on experience, denoted by $x \in \{1, \ldots, X\}$.
- Experience increases by one unit with each period of work, and does not depreciate.
- The preference primitives are given by the current utility from staying home, denoted by $U_{1t}(x_t)$, and the utility from working, $U_{2t}(x_t)$.
- Thus the dynamics of the model come through experience.
- Nonstationarities arise through time varying offer arrival weights, λ_t, and wages (as indicated by t subscripts on current utilities).

Examples of Finite Dependence

Finite dependence in the nonstationary search model

- One period finite dependence is established by constructing two paths; one starts with staying home, $d_{1t} = 1$, the other begins with an employment application, $d_{2t} = 1$.
- Staying home is followed by applying for employment with weight $\lambda_t / \lambda_{t+1}$:

$$\omega_{12t,t+1}(x_t, x_t) = \lambda_t / \lambda_{t+1} = 1 - \omega_{11t,t+1}(x_t, x_t)$$

- Applying for employment is followed by staying home:

$$\omega_{21t,t+1}(x_t, x_t) = \omega_{21t,t+1}(x_t, x_t + 1) = 1$$

- Both sequences generate the same distribution for x_{t+2}:

$$\kappa_{1t1}(x_{t+2}|x_t) = \kappa_{2t1}(x_{t+2}|x_t) = \begin{cases} 1 - \lambda_t \text{ for } x_{t+2} = x_t \\ \lambda_t \text{ for } x_{t+2} = x_t + 1 \end{cases}$$

- Notice that if $\lambda_t > \lambda_{t+1}$ then $\omega_{12t,t+1}(x_t, x_t) > 1$ and $\omega_{11t,t+1}(x_t, x_t) = 1 - \lambda_t / \lambda_{t+1} < 0$.

Determining whether Finite Dependence Exists

Intuition for establishing one period finite dependence in single agent settings

- One period finite dependence holds if there are weights such that:

$$\begin{aligned}
\kappa_{it,t+1}(x_{t+2}|x_t) &\equiv \sum_x \sum_k \omega_{ikt,t+1}(x_t, x) f_{k,t+1}(x_{2+1}|x) f_{it}(x|x_t) \\
&= \sum_x \sum_k \omega_{jkt,t+1}(x_t, x) f_{k,t+1}(x_{2+1}|x) f_{jt}(x|x_t) \\
&\equiv \kappa_{jt,t+1}(x_{t+2}|x_{t+2})
\end{aligned}$$

- Formally this is a linear equation to be solved in the $\omega_{ikt,t+1}(x_t, x)$ and $\omega_{jkt,t+1}(x_t, x)$ terms.
- Some states might not be attainable in $t+1$ from x_t given choice j.
- Denote by $N^*_{t+1}(j, x_t)$ the number of attainable states in period $t+1$ given the choice j in state x_t at time t.
- Denote by $N^*_{t+2}(x_t)$ the number of attainable states in period $t+2$ given *either* initial choice i or j.
- The two choice paths must align the weights on the $N^*_{t+2}(x_t)$ states.

Determining whether Finite Dependence Exists

An algorithm for establishing finite dependence in single agent settings

- Denote $N^*_{\tau+1}(j, x_t)$ as the number of attainable states for a prescribed decision sequence to τ beginning with choice j, and $N^*_{\tau+2}(x_t)$ given either prescribed decision sequence.
- Let $F_{k\tau+1}(j, x_t)$ be a probability transition matrix from each of the $N^*_{\tau+1}(j, x_t)$ attainable states given initial choice j to the $N^*_{\tau+2}(x_t) - 1$ attainable states at $\tau + 2$ given *either* initial choice j or j'.
- Define $\mathcal{F}_{\tau+1}(j, x_t)$ as:

$$\mathcal{F}_{\tau+1}(j, x_t) = \begin{bmatrix} F_{2\tau+1}(j, x_t) - F_{1\tau+1}(j, x_t) \\ \vdots \\ F_{J\tau+1}(j, x_t) - F_{1\tau+1}(j, x_t) \end{bmatrix}^T$$

Theorem (algorithm for establishing finite dependence)

If the rank of $\begin{bmatrix} \mathcal{F}_{\tau+1}(j, x_t) & -\mathcal{F}_{\tau+1}(j', x_t) \end{bmatrix}$ *is* $N^*_{\tau+2}(x+1) - 1$ *then finite dependence can be achieved in* $\tau - t + 1$ *periods.*

Finite Dependence in Games

Example of a coordination game

- Each player $i \in \{1, 2\}$ chooses whether to compete in a market at time t by setting $d_{2t}^{(i)} = 1$ if competing and $d_{1t}^{(i)} = 1$ if not.
- The decisions made by both players in the previous period affect current payoffs, so $x_t = \{d_{2t-1}^{(1)}, d_{2t-1}^{(2)}\}$.
- Nonstationarity arises through the flow payoffs and CCPs.
- This model exhibits two period finite dependence: we find two sequences of choices by the first player, differing by initial the initial choice at t, such that when the second player makes equilibrium choices, the joint distribution of $\left(d_{t+2}^{(1)}, d_{t+2}^{(2)}\right)$ is the same for both sequences:
  1. Choose weights on the decisions at $t + 1$ so that after $t + 2$ the distribution of Player 2's states does not depend on the initial choice.
  2. Set the $t + 2$ choice for Player 1 to be the same across the two paths, guaranteeing his state is the same after the $t + 2$ decision (and has no effect on Player 2's choice at $t + 2$).

Finite Dependence in Games

Some intuition for achieving finite dependence in games

- Establishing finite dependence in games model is more complicated than in single agent models because the decisions of a player today affects the decisions of the other players tomorrow.
- One way of achieving say one period dependence is to first line up the states of the other players through the period $t + 1$ action, and then line up the agent's state at $t + 2$, assuming the agent can line up his own state in one period.
- Intuitively, the choice of one's competitors at $t + 2$ does not depend on the player's choice at $t + 2$ except through their equilibrium expectations over next period's choice conditional on the current state.
- What we would like is that the choice at $t + 2$ of one's competitors lines up the competitors' states at $t + 3$.

Finite dependence

Some notation for achieving finite dependence in games

- Denote $N_{t+3}^{\sim i}$ as all possible competitor states that can result from choice sequences beginning with j or j'.
- Let $\mathcal{F}_{t+1}^{(i)}(j)$ contain the transition probabilities from $t+1$ to $t+2$ given initial choice j by player i.
- Denote $\mathcal{P}_{t+2}^{\sim i}$ as the transpose of the transition matrix from N_{t+2}^{*} feasible period 2 states to the $N_{t+3}^{\sim i} - 1$ competitor states at $t+3$.
- This system of $N_{t+3}^{\sim i} - 1$ equations achieves finite dependence if:
  1. the rank of $\mathcal{P}_{t+2}^{\sim i}\left[\begin{array}{cc} \mathcal{F}_{t+1}^{(i)}(j) & -\mathcal{F}_{t+1}^{(i)}(j') \end{array}\right] = N_{t+3}^{\sim i} - 1$ (because then the competitor states can be lined up at $t+2$)
  2. one's own state can be lined up with the period $t+2$ decision.