# Applying CCP to Dynamic Games

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- Consider a dynamic infinite horizon game for finite $I$ players.
- Thus $T = \infty$ and $I < \infty$.
- Each player $i \in I$ makes a choice $d_t^{(i)} \equiv \left( d_{1t}^{(i)}, \ldots, d_{Jt}^{(i)} \right)$ in period $t$.
- Denote the choices of all the players in period $t$ by:

$$d_t \equiv \left( d_t^{(1)}, \ldots, d_t^{(I)} \right)$$

and denote by:

$$d_t^{(-i)} \equiv \left( d_t^{(1)}, \ldots, d_t^{(i-1)}, d_t^{(i+1)}, \ldots, d_t^{(I)} \right)$$

the choices of $\{1, \ldots, i-1, i+1, \ldots, I\}$ in period $t$, that is all the players apart from $i$.

# A Class of Dynamic Markov Games

## State variables

- Denote by $x_t$ the state variables of the game that are not *iid*.
- For example $x_t$ includes the capital of every firm. Then:
  - firms would have the same state variables.
  - $x_t$ would affect rivals in very different ways.
- We assume all the players observe $x_t$.
- Denote by $F\left(x_{t+1}\left|x_t, d_t\right.\right)$ the probability of $x_{t+1}$ occurs when the state variables are $x_t$ and the players collectively choose $d_t$.
- Similarly let:

$$F_j\left(x_{t+1}\left|x_t, d_t^{(-i)}\right.\right) \equiv F\left(x_{t+1}\left|x_t, d_t^{(-i)}, d_{jt}^{(i)} = 1\right.\right)$$

denote the probability distribution determining $x_{t+1}$ given $x_t$ when $\{1, \ldots, i-1, i+1, \ldots, I\}$ choose $d_t^{(-i)}$ in $t$ and $i$ makes choice $j$.

# A Class of Dynamic Markov Games

## Payoffs and information

- Suppose $\epsilon_t^{(i)} \equiv \left(\epsilon_{1t}^{(i)}, \ldots, \epsilon_{Jt}^{(i)}\right)$, identically and independently distributed with density $g\left(\epsilon_t^{(i)}\right)$, affects the payoffs of $i$ in $t$.
- Also let $\epsilon_t^{(-i)} \equiv \left(\epsilon_t^{(1)}, \ldots, \epsilon_t^{(i-1)}, \epsilon_t^{(i+1)}, \ldots, \epsilon_t^{(I)}\right)$.
- The systematic component of current utility or payoff to player $i$ in period $t$ form taking choice $j$ when everybody else chooses $d_t^{(-i)}$ and the state variables are $z_t$ is denoted by $U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$.
- Denoting by $\beta \in (0, 1)$ the discount factor, the summed discounted payoff to player $i$ throughout the course of the game is:

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} d_{jt}^{(i)} \left[U_j^{(i)}\left(x_t, d_t^{(-i)}\right) + \epsilon_{jt}^{(i)}\right]$$

- Players noncooperatively maximize their expected utilities, moving simultaneously each period. Thus $i$ does not condition on $d_t^{(-i)}$ when making his choice at date $t$, but only sees $\left(x_t, \epsilon_t^{(i)}\right)$.

# Markov Perfect Bayesian Equilibrium

## Markov strategies

- This is a stationary environment and we focus on Markov decision rules, which can be expressed $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$.
- Let $d^{(-i)}\left(x_t, \epsilon_t^{(-i)}\right)$ denote the strategy of every player but $i$:

$$\left(\begin{array}{r} d^{(1)}\left(x_t, \epsilon_t^{(1)}\right), \ldots, d^{(i-1)}\left(x_t, \epsilon_t^{(i-1)}\right), d^{(i+1)}\left(x_t, \epsilon_t^{(i+1)}\right), \\ d^{(i+2)}\left(x_t, \epsilon_t^{(i+2)}\right) \ldots, d^{(I)}\left(x_t, \epsilon_t^{(I)}\right) \end{array}\right)$$

- Then the expected value of the game to $i$ from playing $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$ when everyone else plays $d\left(x_t, \epsilon_t^{(-i)}\right)$ is:

$$V^{(i)}\left(x_1\right) \equiv$$
$$E\left\{\sum_{t=1}^{\infty} \sum_{j=1}^{J} \beta^{t-1} d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)\left[U_j^{(i)}\left(z_t, d\left(x_t, \epsilon_t^{(-i)}\right)\right) + \epsilon_{jt}^{(i)}\right] | x_1\right\}$$

- Integrating over $\epsilon_t^{(i)}$ we obtain the $j^{th}$ conditional choice probability for the $i^{th}$ player at $t$ as $p_j^{(i)}(x_t)$:

$$p_j^{(i)}(x_t) = \int d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right) g\left(\epsilon_t^{(i)}\right) d\epsilon_t^{(i)}$$

- Let $P\left(d_t^{(-i)} | x_t\right)$ denote the joint probability firm $i$'s competitors choose $d_t^{(-i)}$ conditional on the state variables $z_t$.
- Since $\epsilon_t^{(i)}$ is distributed independently across $i \in \{1, \ldots, I\}$:

$$P\left(d_t^{(-i)} | x_t\right) = \prod_{\substack{i'=1 \\ i' \neq i}}^{I} \left(\sum_{j=1}^{J} d_{jt}^{(i')} p_j^{(i')}(x_t)\right)$$

- The strategy $\left\{ d^{(i)} \left( x_t, \epsilon_t^{(i)} \right) \right\}_{i=1}^{I}$ is a Markov perfect equilibrium if, for all $\left( i, x_t, \epsilon_t^{(i)} \right)$, the best response of $i$ to $d^{(-i)} \left( x_t, \epsilon_t^{(-i)} \right)$ is $d^{(i)} \left( x_t, \epsilon_t^{(i)} \right)$ when everybody uses the same strategy thereafter.
- That is, suppose the other players collectively use $d^{(-i)} \left( x_t, \epsilon_t^{(-i)} \right)$ in period $t$, and $V^{(i)} (x_{t+1})$ is formed from $\left\{ d^{(i)} \left( x_t, \epsilon_t^{(i)} \right) \right\}_{i=1}^{I}$.
- Then $d^{(i)} \left( x_t, \epsilon_t^{(i)} \right)$ solves for $i$ choosing $j$ to maximize:

$$\sum_{d_t^{(-i)}} P \left( d_t^{(-i)} | x_t \right) \left\{ \begin{array}{l} U_j^{(i)} \left( x_t, d_t^{(-i)} \right) \\ + \beta \sum_{z=1}^{X} V^{(i)} (x) F_j \left( x \middle| x_t, d_t^{(-i)} \right) \end{array} \right\} + \epsilon_{jt}^{(i)}$$

# Adapting Dynamic Games to the CCP Framework

## Connection to Individual Optimization

- In equilibrium, the systematic component of the current utility of player $i$ in period $t$, as a function of $x_t$, the state variables for game, and his own decision $j$, is:

$$u_j^{(i)}(x_t) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} \left| x_t\right.\right) U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$$

- Similarly the probability transition from $x_t$ to $x_{t+1}$ given action $j$ by firm $i$ is given by:

$$f_j^{(i)}\left(x_{t+1}\left|x_t^{(i)}\right.\right) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)}\left|x_t^{(i)}\right.\right) F_j\left(x_{t+1}\left|x_t, d_t^{(-i)}\right.\right)$$

- The setup for player $i$ is now identical to the optimization problem described in the second lecture for a stationary environment.

- Note that:
  1. there might be multiple equilibria, but we assume:
     - either every firm plays in the same market
     - or every market plays the same equilibrium.
  2. in contrast to ML we do not solve for the equilibrium.
  3. estimation is based on conditions that are satisfied by every Markov perfect equilibrium.
  4. the estimation approach is identical to the approach we described in the individual optimization problem.
- Thus the basic difference between estimating this dynamic game and an individual optimization problem using a CCP estimator revolves around how much the payoffs of each player are affected by state variables partially determined by other players through their conditional choice probabilities.

# Entry Exit Game

## Choice Variables

- Suppose there is a finite maximum number of firms in a market at any one time denoted by $I$.
- If a firm exits, the next period an opening occurs to a potential entrant, who may decide to exercise this one time option, or stay out.
- At the beginning of each period every incumbent firm has the option of quitting the market or staying one more period.
- Let $d_t^{(i)} \equiv \left( d_{1t}^{(i)}, d_{2t}^{(i)} \right)$, where $d_{1t}^{(i)} = 1$ means $i$ exits or stays out of the market in period $t$, and $d_{2t}^{(i)} = 1$ means $i$ enters or does not exit.
- If $d_{2t}^{(i)} = 1$ and $d_{1,t-1}^{(i)} = 1$ then the firm in spot $i$ at time $t$ is an entrant, and if $d_{2,t-1}^{(i)} = 1$ the spot $i$ at time $t$ is an incumbent.

# Entry Exit Game

## State Variables

- In this application there are three components to the state variables and $x_t = (x_1, x_{2t}, s_t)$.
- The first is a permanent market characteristic, denoted by $x_1$, and is common across firms in the market. Each market faces an equal probability of drawing any of the possible values of $x_1$ where $x_1 \in \{1, 2, \ldots, 10\}$.
- The second, $x_{2t}$, is whether or not each firm is an incumbent, $x_{2t} \equiv \{d_{2t-1}^{(1)}, \ldots, d_{2t-1}^{(I)}\}$. Entrants pay a start up cost, making it more likely that stayers choose to fill a slot than an entrant.
- A demand shock $s_t \in \{1, \ldots, 5\}$ follows a first order Markov chain.
- In particular, the probability that $s_{t+1} = s_t$ is fixed at $\pi \in (0, 1)$, and probability of any other state occurring is equally likely:

$$\Pr\{s_{t+1} | s_t\} = \begin{cases} \pi \text{ if } s_{t+1} = s_t \\ (1-\pi)/4 \text{ if } s_{t+1} \neq s_t \end{cases}$$

- Each active firm produces one unit so revenue, denoted by $y_t$, is just price.
- Price is determined by:
  1. the supply of active firms in the market, $\sum_{i=1}^{I} d_{2t}^{(i)}$
  2. a permanent market characteristic, $x_1$
  3. the Markov demand shock $s_t$
  4. another temporary shock, denoted by $\eta_t$, distributed *iid* standard normal distribution, revealed to each market after the entry and exit decisions are made.
- The price equation is:

$$y_t = \alpha_0 + \alpha_1 x_1 + \alpha_2 s_t + \alpha_3 \sum_{i=1}^{I} d_{2t}^{(i)} + \eta_t$$

# Entry Exit Game

## Expected Profits conditional on competition

- We assume costs comprise a choice specific disturbance $\epsilon_{jt}^{(i)}$ that is privately observed, plus a linear function of $z_t$.
- Net current profits for exiting incumbent firms, and potential entrants who do not enter, are $\epsilon_{1t}^{(i)}$. Thus $U_1^{(i)}\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) \equiv 0$.
- Current profits from being active are the sum of $\left(\epsilon_{2t}^{(i)} + \eta_t\right)$ and:

$$U_2^{(i)}\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) \equiv \theta_0 + \theta_1 x_1 + \theta_2 s_t + \theta_3 \sum_{\substack{i'=1 \\ i' \neq i}}^{I} d_{2t}^{(i')} + \theta_4 d_{1,t-1}^{(i)}$$

  where $\theta_4$ is the startup cost that only entrants pay.
- In equilibrium $E\left(\eta_t\right) = 0$ so:

$$u_j^{(i)}\left(x_t, s_t\right) = \theta_0 + \theta_1 x_1 + \theta_2 s_t + \theta_3 \sum_{\substack{i'=1 \\ i' \neq i}}^{I} p_2^{(i')}\left(x_t, s_t\right) + \theta_4 d_{1,t-1}^{(i)}$$

# Entry Exit Game

## Terminal Choice Property

- We assume the firm's private information, $\epsilon_{jt}^{(i)}$, is distributed Type 1 extreme value.
- Since exiting is a terminal choice, with the exit payoff normalized to zero, the Type 1 extreme value assumption implies that the conditional value function for being active is:

$$
\begin{aligned}
v_2^{(i)}(x_t, s_t) &= u_2^{(i)}(x_t, s_t) \\
&\quad -\beta \sum_{x \in X} \sum_{s \in S} \left( \ln \left[ p_1^{(i)}(x, s) \right] \right) f_2^{(i)}(x, s | x_t, s_t)
\end{aligned}
$$

- The future value term is then expressed as a function solely of the one-period-ahead probabilities of exiting and the transition probabilities of the state variables.

- The number of firms in each market is set to six and we simulated data for 3,000 markets.
- The discount factor is set at $\beta = 0.9$.
- Starting at an initial date with six potential entrants in the market, we solved the model, ran the simulations forward for twenty periods, and used the last ten periods to estimate the model.
- The key difference between this Monte Carlo and the renewal Monte Carlo is that the conditional choice probabilities have an additional effect on both current utility and the transitions on the state variables due to the effect of the choices of the firm's competitors on profits.

# Entry Exit Game

## Extract from Table 2 of Arcidiacono and Miller (2011)

Monte Carlo for Entry/Exit Game⁺

| | | DGP (1) | CCP (2) |
|---|---|---|---|
| Profit Parameters | $\theta_0$ (Intercept) | 0 | 0.0207 (0.0779) |
| | $\theta_1$ (Market) | 0.05 | -0.0505 (0.0028) |
| | $\theta_2$ (Demand) | 0.25 | 0.2529 (0.0080) |
| | $\theta_3$ (No. of Competitors) | -0.2 | -0.2061 (0.0207) |
| | $\theta_4$ (Entry cost) | -1.5 | -1.4992 (0.0131) |
| Price Parameters | $\alpha_0$ (Intercept) | 7 | 6.9973 (0.0296) |
| | $\alpha_1$ (Market) | -0.1 | -0.0998 (0.0023) |
| | $\alpha_2$ (Demand) | 0.3 | 0.2996 (0.0045) |
| | $\alpha_3$ (No. of Competitiors) | -0.4 | -0.3995 (0.0061) |
| | $\pi$ (Persistence of Unobs. State) | 0.7 | |
| | Time (Minutes) | | 0.1354 (0.0047) |

⁺ Mean and standard deviations for 100 simulations. Observed data consist of 3000 markets for 10 periods with 6 firms in each market.