# Bidder Valuations

Robert A. Miller

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- In first price sealed bid (FPSB) auctions the highest bidder wins and pays his bid.
- In second price sealed bid auctions (SPSB) the highest bidder wins and pays the bid of highest losing bidder.
- In Dutch auctions (reducing the price until a player accepts the offer) only the winning bid is ever observed; Dutch auctions are strategically equivalent to FPSB auctions.
- In Japanese (button) auctions players exit as the auctioneer raises the price and the winner pays the price at which the only other remaining bidder exits.
- Note that players update their information sets in Japanese auctions so are not necessarily strategically equivalent to SPSB auctions.

# Example

## Independent and identically distributed private values in a first price sealed bid auction

- We first consider a first price sealed bid (FPSB) auction for $N$ players with independent private values (IPV).
- By FPSB we mean that each player $n \in \{1, \ldots, N\}$ simultaneously submits a bid denoted by $b_n \in \mathbf{R}^+$, and that the player submitting the highest bidder is awarded the (single) object up for auction, and pays what he or she bid.
- By IPV we mean that for each $n \in \{1, \ldots, N\}$ the value of owning the object is $v_n$ where $v_n \in \mathbf{V}$ independently drawn from a common distribution, $F(v)$.

# Example

## Best replies in equilibrium

- Let $W(b)$ denote the probability of winning the auction with bid $b$. That is:

$$W(b) \equiv \Pr\{b_k \leq b \text{ for all } k = 1, \ldots, N\}$$

- Then the maximization problem faced by player $n$ can be written as:

$$\max_b (v_n - b) W(b)$$

- The first order condition (FOC) is:

$$(v_n - b_n) W'(b_n) - W(b_n) = 0 \tag{1}$$

- The second order condition (SOC) of the optimization problem is:

$$\begin{aligned} \frac{\partial}{\partial b} FOC &\equiv \frac{\partial}{\partial b}\left[(v - b) W'(b) - W(b)\right] \\ &= (v - b) W''(b) - 2W'(b) \\ &\equiv SOC \\ &< 0 \end{aligned}$$

# Example

## Pure strategy best replies are increasing in valuations

- Totally differentiating the FOC with respect to $b$ and $v$ yields:

$$0 = W'(b_n)\, dv_n + \left[ (v_n - b_n) W''(b_n) - 2W'(b_n) \right] db_n$$

and hence:

$$\frac{db_n}{dv_n} = \frac{-W'(b_n)}{(v_n - b_n) W''(b_n) - 2W'(b_n)} > 0$$

because $W'(b_n) > 0$ and the denominator of the quotient is the SOC.

- We infer that if players are in a pure strategy equilibrium with an interior solution, then $b_n$ is increasing in $v_n$.

# Example

## Bayesian Nash Equilibrium with monotone bidding

- From now on we assume that players are in a (pure strategy) Bayesian equilibrium with bids that are monotone increasing in valuations.
- That is we consider Bayesian Nash Equilibrium (BNE) in which bidders follow a strategy $\beta : \mathbf{V} \rightarrow \mathbf{B} \equiv [0, \infty)$ where $\beta(v)$ is increasing in $v$.
- Then $\beta(v)$ has an inverse, which we denote by $\alpha : \mathbf{B} \rightarrow \mathbf{V}$ such that $\alpha[\beta(v)] = v$ for all $v$.
- Letting $G(b)$ denote the distribution of bids, it follows that:

$$W(b) \equiv \Pr\{b_k \leq b_n \text{ for all } k = 1, \ldots, N\} = G(b_n)^{N-1}$$

- From the monotonicity property of the BNE:

$$G(b) = F(\alpha(b))$$

# Example

## Identification when all bids are observed from the probability of winning

- Assume our data set consist of all the bids recorded in $I$ auctions in which the same equilibrium is played.
- Let $b_n^i$ for $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, I\}$ denote the $n^{th}$ bid in the $i^{th}$ auction.
- The probability of winning the auction, $W(b)$, and its derivative $W'(b)$ are identified.
- We rewrite the FOC, Equation (1) as:

$$v_n^i = b_n^i + \frac{W\left(b_n^i\right)}{W'\left(b_n^i\right)} \tag{2}$$

- This shows $v_n^i$ is identified, and therefore so is $F(v)$.

# Example

## A nonparametric estimator of the valuations

- Suppose there are a total of $NI$ bids in the $I$ auctions.
- Then $v_n^i$ can be estimated nonparametrically with:

$$v_n^i = b_n^i + \widehat{W}\left(b_n^i\right) \bigg/ \frac{\partial \widehat{W}\left(b_n^i\right)}{\partial b} \tag{3}$$

where:

$$\widehat{W}(b) = \frac{\sum_{m=1}^{M \equiv NI} I\left\{b_m \text{ is a winning bid}\right\} K\left(\frac{b-b_m}{h_M}\right)}{\sum_{m=1}^{M \equiv NI} K\left(\frac{b-b_m}{h_M}\right)}$$

and appealing to Collomb (1977; C.R. Acad. Sci. Paris 285, 289-292):

- $K(\cdot)$ is uniformly continuous, has bounded variation, and is integrable, with $K(-\infty) = K(\infty) = 0$ and $0 < K(0) < \infty$;
- for example $K(\cdot)$ is a standard normal pdf;
- $h_M \to 0$ and $M^{-1} h_M^{-1} \log M \to 0$ as $M \to \infty$.

# Example

## Identification when all bids are observed from the bidding distribution

- Alternatively note that the probability distribution of bids and its density, $G(b)$ and $G'(b)$, are identified.
- But the probability $n$ wins with $b_n$ is:

$$W(b_n) = G(b_n)^{N-1}$$

implying

$$W'(b_n) = (N-1)\, G(b_n)^{N-2} G'(b_n)$$

- We rewrite the FOC, Equation $(1)$ as:

$$v_n^i = b_n^i + \frac{W(b_n^i)}{W'(b_n^i)} = b_n^i + \frac{G(b_n^i)}{(N-1)\, G'(b_n^i)} \tag{4}$$

- This shows $v_n^i$ and hence $F(v)$ can also be directly identified off the bidding distribution $G(b)$.

# Example

## The distribution of winning bids

- Now suppose our data set consist of only the winning bid recorded in $I$ auctions in which the same equilibrium is played.
- Let $b^i$ for $i \in \{1, \ldots, I\}$ denote the winning bid in the $i^{th}$ auction.
- Thus the distribution of winning bids, denoted by $H\left(b^i\right)$, is identified.
- Since the winning bid is defined as the highest one, $H(b)$ is just the probability that all the bids are less than $b$, implying:

$$H(b) = \Pr\left\{b_n^i \leq b \text{ for all } n = 1, \ldots, N\right\} = G(b)^N$$

- Consequently:

$$G(b) = H(b)^{\frac{1}{N}} \tag{5}$$

and

$$G'(b) = \frac{1}{N} H(b)^{\frac{1}{N}-1} H'(b) \tag{6}$$

- This shows the bidding distribution is identified from the data generating process of the winner's bid.

# Example

## Identification when only the winning bid is observed

- Substituting Equations (5) and (6) back into Equation (4) gives:

$$v^i = b^i + \frac{G\left(b^i\right)}{(N-1)\,G'\left(b^i\right)} = b^i + \frac{NH(b)}{(N-1)\,H'(b)}$$

- This identifies the winning valuations, and hence their distribution, denoted by $F_W(v)$.
- But the distribution of the winning valuations is a one to one mapping of the distribution of all the valuations:

$$F_W(v) = \Pr\{v_n \leq v \text{ for all } n = 1, \ldots, N\} = F(v)^N$$

- Therefore $F(v)$ is identified off the winning bids alone using the equation:

$$F(v) = F_W(v)^{\frac{1}{N}}$$

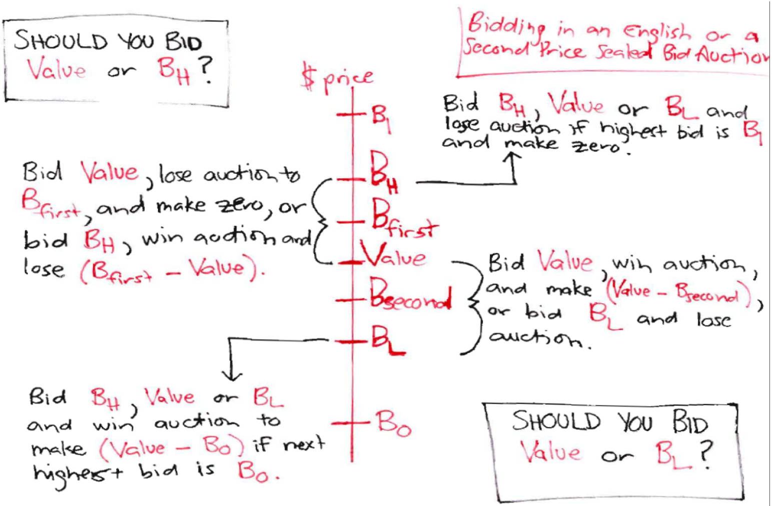
# Another Example

## A second price sealed bid (SPSB) auction with private values

- Now suppose as before:
  - each bidder knows her own valuation;
  - makes sealed bid (that is bids simultaneously).
- But instead of a FPSB auction, consider a SPSB auction, where the highest bidder wins the auction but only pays the second highest bid.
- Now it is a weakly dominant strategy for (each) $n$ to bid her expected valuation, $v_n$.
- Intuitively, compared with bidding $v_n$:
  - bidding more implies winning some auctions that yield negative expected value, but leaves unchanged the expected value of any other auction that would be won;
  - bidding less implies losing some auctions that yield positive expected value, but leaves unchanged the expected value of any other auction that she would win.

# Another Example

## A picture proof



# Another Example

## Distribution of the second highest valuation

- Let $F(v)$ denote the distribution of valuations as before.
- Note first the obvious point that because players bid their valuations in SPSB auctions with private valuations, $F(v)$ is trivially identified if all the bids are observed.
- Now suppose only the winning price is observed.
- Then the probability distribution of the second highest valuation, which we now denote by $F_{N-1,N}(v)$, is identified.

# Another Example

## Identification when only the winning bid is observed

- More generally, let $F_{i,N}(v)$ denote the distribution of the $i^{th}$ order statistic.
- Then it can be shown (see Arnold, Balakrishnan and Nagaraja,1992 for example) that:

$$F_{i,N}(v) = \frac{N!}{(N-i)!\,(i-1)!} \int_{\underline{v}}^{F(v)} t^{i-1} (1-t)^{N-i}\, dt$$

- Note that $\underline{v}$ is identified (and a consistent estimate is the lowest winning bid observed in the data).
- Also:

$$\frac{\partial F_{i,N}(v)}{\partial v} = \frac{N!}{(N-i)!\,(i-1)!} F(v)^{i-1} \left[1 - F(v)\right]^{N-i}$$

- There are $N$ players, risk neutral bidders.
- Bidder $n$:
  - has valuation $v_n$, the utility gain from winning the auction.
  - receives signal $x_n$, which without loss of generality we normalize, setting $x_n \equiv E\left[v_n \,|x_n\right]$. Let $x \equiv (x_1, \ldots, x_N)$.
- We often assume $y \equiv (v_1, \ldots, v_N, x)$ is affiliated, higher realizations of one component associated with higher realizations of the others.
- This means for random variable $Y$ with *pdf* $f_Y(y)$, where $\vee$ $(\wedge)$denotes the component wise maximum (minimum):

$$f_Y\left(y \vee y'\right) f_Y\left(y \wedge y'\right) \geq f_Y(y)\, f_Y\left(y'\right)$$

- Bidders have:
  - private valuations if $E\left[v_n \,|x\right] = x_n$;
  - common valuations if $E\left[v_n \,|x\right]$ is strictly increasing in all its conditioning components $(x_1, \ldots, x_N)$.
  - pure common values if $E\left[v_n \,|x\right] = E\left[v_{n'} \,|x\right]$ for all $n$ and $n'$.

- The literature focuses on perfect Bayesian equilibria in weakly undominated pure strategies (Athey and Haile, 2006).
- Let $b_n \equiv \beta_n(x_n, N)$ denote the equilibrium strategy of bidder $n$.
- In a second price auction with private values, it is a weakly dominant strategy for (each) $n$ to bid his expected valuation, setting:

$$\beta_n(x_n, N) = x_n \equiv E[v_n | x_n]$$

- Note the same logic applies to $n$ individually if $v_n = x_n$, regardless of the correlation structure of $y$ and the other bidders' information.

- In a private value FPSB auction denote the bid distribution function for the maximum equilibrium bid of the $n^{th}$ bidder's rivals, conditional on the signal of $n$, as:

$$G_{m_n}(m | x_n, N) = \Pr\left[\max_{n' \in N \backslash n} \{b_{n'}\} \leq m | x_n, N\right]$$

- Then $b_n$ solves:

$$b_n = \arg\max_b \int_{-\infty}^{b} (x_n - b)\, G'_{m_n}(m | x_n, N)\, dm$$

- The first order condition is:

$$x_n = b_n + \frac{G_{m_n}(b_n | x_n, N)}{G'_{m_n}(b_n | x_n, N)}$$

- Note this FOC reduces to (2) when $v_n = x_n$ and the valuations of the bidders are *iid*; in any case both $W(b_n)$ and $G_{m_n}(b_n | x_n, N)$ represent the probability of $n$ winning the auction with bid $b_n$.

- At a superficial level, this first order condition takes a similar form in a common value auction. Define:

$$v_n\left(x_n, x_n', N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \setminus n}\{b_{n'}\} = \beta_n\left(x_n', N\right)\right]$$

- Similar to the private values case $b_n$ solves:

$$b_n = \arg\max_b \int_{-\infty}^{b}\left[v_n\left(x_n, \beta_n^{-1}(m, N), N\right) - b\right] G'_{m_n|b}(m\,|x_n, N)\,dm$$

- The first order condition is:

$$v_n\left(x_n, x_n, N\right) = b_n + \frac{G_{m_n|b}(b_n\,|x_n, N)}{G'_{m_n|b}(b_n\,|x_n, N)}$$

# Identification in FPSB Auctions with Private Values

## When all the bids are observed

- Assume $x_n = v_n$. From the first order condition:

$$x_n = b_n + \frac{G_{m_n}(b_n | x_n, N)}{G'_{m_n}(b_n | x_n, N)}$$

- Recall from its definition that $G_{m_n}(b_n | x_n, N)$ is the probability that $n$ wins the auction with $b_n$:

$$G_{m_n}(b_n | x_n, N) = \Pr\left[\max_{n' \in N \backslash n} \{b_{n'}\} \leq b_n | x_n, N\right]$$

- Thus if all the bids are observed then $G_{m_n}(b_n | x_n, N)$ is identified.
- Hence $v_n$ is identified (for all bidders in each sampled auction).
- Therefore the probability distribution of $(v_1, \ldots, v_N)$ in this specialization is identified for any correlation structure.

# Identification Fails in Common Value FPSB Auctions

## When all the bids are observed

- Recall that we defined:

$$v_n\left(x_n, x_n', N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \backslash n}\left\{b_{n'}\right\} = \beta_n\left(x_n', N\right)\right]$$

and derived:

$$v_n\left(x_n, x_n, N\right) = b_n + \frac{G_{m_n}\left(b_n \left| x_n, N\right.\right)}{G_{m_n}'\left(b_n \left| x_n, N\right.\right)}$$

- The basic problem is that conditional on $N$ the RHS gives a number for each $n$, but the LHS is not a primitive of the model.
- Note that every common value model is observationally to a private value model found by setting $v_n = v_n\left(x_n, x_n', N\right)$.
- Thus two common value models with the same $v_n\left(x_n, x_n', N\right)$ are (also) observationally equivalent.