# Sealed Bid Auctions

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- In first price sealed bid (FPSB) auctions the highest bidder wins and pays his bid.
- In second price sealed bid auctions (SPSB) the highest bidder wins and pays the bid of highest losing bidder.
- In Dutch auctions (reducing the price until a player accepts the offer) only the winning bid is ever observed; Dutch auctions are strategically equivalent to FPSB auctions.
- In Japanese (button) auctions players exit as the auctioneer raises the price and the winner pays the price at which the only other remaining bidder exits.
- Note that players update their information sets in Japanese auctions so are not necessarily strategically equivalent to SPSB auctions.

# Example

## Independent and identically distributed private values in a first price sealed bid auction

- We first consider a first price sealed bid (FPSB) auction for $N$ players with independent private values (IPV).
- By FPSB we mean that each player $n \in \{1, \ldots, N\}$ simultaneously submits a bid denoted by $b_n \in \mathbf{R}^+$, and that the player submitting the highest bid is awarded the (single) object up for auction, and pays what he or she bid.
- By IPV we mean that for each $n \in \{1, \ldots, N\}$ the value of owning the object is $v_n$ where $v_n \in \mathbf{V}$ independently drawn from a common distribution, $F(v)$.

# Example

## Best replies in equilibrium

- Let $W(b)$ denote the probability of winning the auction with bid $b$. That is:
$$W(b) \equiv \Pr\{b_k \leq b \text{ for all } k = 1, \ldots, N\}$$
- Then the maximization problem faced by player $n$ can be written as:
$$\max_b (v_n - b) W(b)$$
- The first order condition (FOC) is:
$$(v_n - b_n) W'(b_n) - W(b_n) = 0 \tag{1}$$
- The second order condition (SOC) of the optimization problem is:
$$\begin{aligned} 0 > SOC \equiv \frac{\partial}{\partial b} FOC &= \frac{\partial}{\partial b}\left[(v - b) W'(b) - W(b)\right] \\ &= (v - b) W''(b) - 2W'(b) \end{aligned}$$

# Example

## Pure strategy best replies are increasing in valuations

- Totally differentiating the FOC with respect to $b$ and $v$ yields:

$$0 = W'(b_n)\, dv_n + \left[(v_n - b_n) W''(b_n) - 2W'(b_n)\right] db_n$$

and hence:

$$\frac{db_n}{dv_n} = \frac{-W'(b_n)}{(v_n - b_n) W''(b_n) - 2W'(b_n)} > 0$$

because $W'(b_n) > 0$ and the denominator of the quotient is the SOC.

- We infer that if players are in a pure strategy equilibrium with an interior solution, then $b_n$ is increasing in $v_n$.

- From now on we assume that players are in a (pure strategy) Bayesian equilibrium with bids that are monotone increasing in valuations.
- That is we consider Bayesian Nash Equilibrium (BNE) in which bidders follow a strategy $\beta : \mathbf{V} \rightarrow \mathbf{B} \equiv [0, \infty)$ where $\beta(v)$ is increasing in $v$.
- Then $\beta(v)$ has an inverse, which we denote by $\alpha : \mathbf{B} \rightarrow \mathbf{V}$ such that $\alpha[\beta(v)] = v$ for all $v$.
- Letting $G(b)$ denote the distribution of bids, it follows that:

$$W(b) \equiv \Pr\{b_k \leq b_n \text{ for all } k = 1, \dots, N\} = G(b_n)^{N-1}$$

- From the monotonicity property of the BNE:

$$G(b) = F(\alpha(b))$$

# Example

## Identification when all bids are observed from the probability of winning

- Assume our data set consists of all the bids recorded in $I$ auctions in which the same equilibrium is played.
- Let $b_n^i$ for $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, I\}$ denote the bid by player $n$ in the $i^{th}$ auction.
- The probability of winning the auction, $W(b)$, and its derivative $W'(b)$ are identified.
- We rewrite the FOC, Equation (1) as:

$$v_n^i = b_n^i + \frac{W\left(b_n^i\right)}{W'\left(b_n^i\right)} \tag{2}$$

- This shows $v_n^i$ is identified, and therefore so is $F(v)$.

# Example

## Identification when all bids are observed from the bidding distribution

- Alternatively note that the probability distribution of bids and its density, $G(b)$ and $G'(b)$, are identified.
- But the probability $n$ wins with $b_n$ is:

$$W(b_n) = G(b_n)^{N-1}$$

implying

$$W'(b_n) = (N-1)\, G(b_n)^{N-2} G'(b_n)$$

- We rewrite the FOC, Equation (1) as:

$$v_n^i = b_n^i + \frac{W(b_n^i)}{W'(b_n^i)} = b_n^i + \frac{G(b_n^i)}{(N-1)\, G'(b_n^i)} \tag{3}$$

- This shows $v_n^i$ and hence $F(v)$ can also be directly identified off the bidding distribution $G(b)$.

# Example
## The distribution of winning bids

- Now suppose our data set consists of only the winning bid recorded in $I$ auctions in which the same equilibrium is played.
- Let $b^i$ for $i \in \{1, \ldots, I\}$ denote the winning bid in the $i^{th}$ auction.
- Thus the distribution of winning bids, denoted by $H\left(b^i\right)$, is identified.
- Since the winning bid is defined as the highest one, $H(b)$ is just the probability that all the bids are less than $b$, implying:

$$H(b) = \Pr\left\{b_n^i \leq b \text{ for all } n = 1, \ldots, N\right\} = G(b)^N$$

- Consequently:

$$G(b) = H(b)^{\frac{1}{N}} \tag{4}$$

and

$$G'(b) = \frac{1}{N} H(b)^{\frac{1}{N}-1} H'(b) \tag{5}$$

- This shows the bidding distribution is identified from the data generating process of the winner's bid.

# Example

## Identification when only the winning bid is observed

- Substituting Equations (4) and (5) back into Equation (3) gives:

$$v^i = b^i + \frac{G\left(b^i\right)}{(N-1)\, G'\left(b^i\right)} = b^i + \frac{NH\left(b\right)}{(N-1)\, H'\left(b\right)}$$

- This identifies the winning valuations, and hence their distribution, denoted by $F_W(v)$.
- But the distribution of the winning valuations is a one to one mapping of the distribution of all the valuations:

$$F_W(v) = \Pr\left\{v_n \leq v \text{ for all } n = 1, \ldots, N\right\} = F(v)^N$$

- Therefore $F(v)$ is identified off the winning bids alone using the equation:

$$F(v) = F_W(v)^{\frac{1}{N}}$$

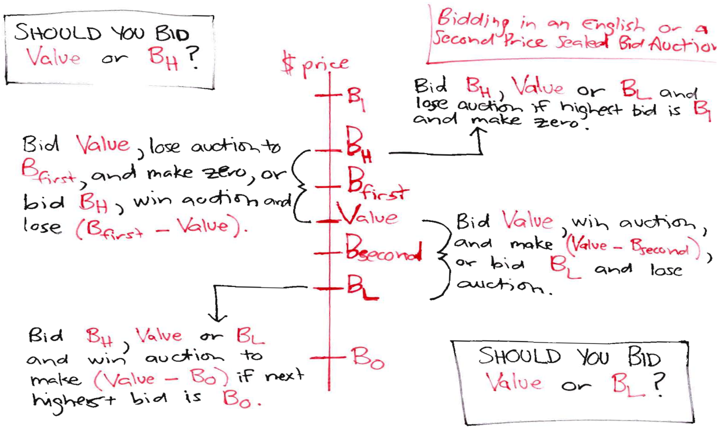
# Another Example

## A second price sealed bid (SPSB) auction with private values

- Now suppose as before:
  - each bidder knows her own valuation;
  - makes  sealed bid (that is bids simultaneously).
- But instead of a FPSB auction, consider a SPSB auction, where the highest bidder wins the auction but only pays the second highest bid.
- Now it is a weakly dominant strategy for (each) $n$ to bid her expected valuation, $v_n$.
- Intuitively, compared with bidding $v_n$:
  - bidding more implies winning some auctions that yield negative expected value, but leaves unchanged the expected value of any other auction that would be won;
  - bidding less implies losing some auctions that yield positive expected value, but leaves unchanged the expected value of any other auction that she would win.

# Another Example

## A picture proof



# Another Example

## Distribution of the second highest valuation

- Let $F(v)$ denote the distribution of valuations as before.
- Note first the obvious point that because players bid their valuations in SPSB auctions with private valuations, $F(v)$ is trivially identified if all the bids are observed.
- Now suppose only the winning price is observed.
- Then the probability distribution of the second highest valuation, which we now denote by $F_{N-1,N}(v)$, is identified.

# Another Example

## Distribution of the second highest valuation

- More generally, let $F_{i,N}(v)$ denote the distribution of the $i^{th}$ order statistic. Note that:
  - The probability that the first $i-1$ draws are less than $v$ and the next $N-i$ are greater than $v$ is:

$$\int_{\underline{v}}^{F(v)} t^{i-1}(1-t)^{N-i}\,dt$$

  - The number of permutations with exactly $i-1$ draws less than $v$ from $N-1$ draws is:

$$\binom{N-1}{i} = \frac{(N-1)!}{(N-i)!\,(i-1)!}$$

  - Any one of $N$ draws can be the $i^{th}$ highest valuation.
- Therefore:

$$F_{i,N}(v) = \frac{N!}{(N-i)!\,(i-1)!}\int_{\underline{v}}^{F(v)} t^{i-1}(1-t)^{N-i}\,dt \tag{6}$$

# Another Example

## Identification of the probability distribution of valuations

- Clearly $\underline{v}$ is identified, because a consistent estimate of $\underline{v}$ is the lowest winning payment observed in the data.
- We now show by a contradiction argument the mapping from $F_{i,N}(v)$ to $F(v)$ is invertible.
- Suppose there are two (or more solutions) solutions to (6):
  - Denote them by by $F_1(v)$ and $F_2(v)$.
  - Substitute $F_i(v)$ into (6) for $i \in \{1, 2\}$.
  - Difference the two resulting equations.
  - Divide through by $N! \big/ (N-i)!\,(i-1)!$ to obtain:

$$\int_{\underline{v}}^{F_1(v)} t^{i-1} (1-t)^{N-i}\, dt = \int_{\underline{v}}^{F_2(v)} t^{i-1} (1-t)^{N-i}\, dt$$

  - Since $t^{i-1}(1-t)^{N-i} > 0$ it immediately follows that $F_1(v) = F_2(v)$.

# Theoretical Foundations

## Notation and terminology for sealed bid auctions

- There are $N$ risk neutral bidders. Bidder $n$:
  - has valuation $v_n$, the utility gain from winning the auction.
  - receives signal $x_n \equiv v_n + \epsilon_n$, where $E[\epsilon_n | x_n] = 0$.
- Denote $x \equiv (x_1, \ldots, x_N)$ and $v \equiv (v_1, \ldots, v_N)$ and $y \equiv (v, x)$.
- We often assume $y$ is affiliated, higher realizations of one component associated with higher realizations of the others.
- This means for random variable $Y$ with *pdf* $f_Y(y)$, where $\vee$ $(\wedge)$denotes the component wise maximum (minimum):

$$f_Y(y \vee y') f_Y(y \wedge y') \geq f_Y(y) f_Y(y')$$

- Noting $x_n \equiv E[v_n | x_n]$, we say bidders have:
  - private valuations if $E[v_n | x] = x_n$;
  - common valuations if $E[v_n | x_1, \ldots, x_N]$ is strictly increasing in all $x_m \in \{x_1, \ldots, x_N\}$.
  - pure common values if $E[v_m | x] = E[v_n | x]$ for all $m$ and $n$.

# Theoretical Foundations

## Affiliation

- If $f_Y(y) > 0$ and twice differentiable then affiliation is equivalent to:

$$\partial f_Y(y) / \partial y_n \partial y_m \geq 0$$

- Also if $Y_n$ and $Y_m$ are affiliated, then for all $y_n \geq y_n'$ and $y_m \geq y_m'$:

$$\begin{aligned}
f_Y(y_n, y_m) f_Y(y_n', y_m') &\geq f_Y(y_n, y_n') f_Y(y_m, y_m') \\
\iff \frac{f(y_n | y_m)}{f(y_n | y_n')} f(y_m) f(y_n') &\geq \frac{f(y_m' | y_m)}{f(y_m' | y_n')} f(y_m) f(y_n') \\
\iff \frac{f(y_n | y_m)}{f(y_n | y_n')} &\geq \frac{f(y_m' | y_m)}{f(y_m' | y_n')}
\end{aligned}$$

- In words the CDF $F(y | y_m)$ dominates $F(y | y_n')$ in terms of the likelihood ratio and hence one can show:
  - $F(y | y_m)$ first order dominates $F(y | y_n')$.
  - the likelihood ratio $f(y | y_m) / f(y | y_n')$ is increasing in $y$.

# Theoretical Foundations

## Equilibrium best responses in second price auctions with private values

- The literature focuses on perfect Bayesian equilibria in weakly undominated pure strategies (Athey and Haile, 2006).
- Let $b_n \equiv \beta_n(x_n, N)$ denote the equilibrium strategy of bidder $n$.
- In a second price auction with private values, it is a weakly dominant strategy for (each) $n$ to bid his expected valuation, setting:

$$\beta_n(x_n, N) = x_n \equiv E[v_n | x_n]$$

- Note the same logic applies to $n$ individually if $v_n = x_n$, regardless of the correlation structure of $y$ and the other bidders' information.

# Theoretical Foundations

## Equilibrium best responses in first price auctions with private values

- In a private value FPSB auction denote the CDF for the maximum equilibrium bid of the $n^{th}$ bidder's rivals, conditional on the signal of $n$, by:

$$G_{m_n}\left(b_m \left| x_n, N\right.\right) = \Pr\left[\max_{n' \in N \backslash n}\left\{b_{n'}\right\} \leq b_m \left| x_n, N\right.\right]$$

- Then $b_n$ solves:

$$b_n = \arg\max_b \int_{-\infty}^{b} \left(x_n - b\right) G'_{m_n}\left(b_m \left| x_n, N\right.\right) db_m$$

- The first order condition is:

$$x_n = b_n + \frac{G_{m_n}\left(b_n \left| x_n, N\right.\right)}{G'_{m_n}\left(b_n \left| x_n, N\right.\right)}$$

- Note this FOC reduces to (2) when $v_n = x_n$ and the valuations of the bidders are *iid*; in any case both $W\left(b_n\right)$ and $G_{m_n}\left(b_n \left| x_n, N\right.\right)$ represent the probability of $n$ winning the auction with bid $b_n$.

- At a superficial level, this first order condition takes a similar form in a common value auction. Define:

$$v_n\left(x_n, x_{n'}, N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \setminus n} \left\{b_{n'}\right\} = \beta_n\left(x_{n'}, N\right)\right]$$

- Similar to the private values case $b_n$ solves:

$$b_n = \arg\max_b \int_{-\infty}^{b} \left[v_n\left(x_n, \beta_n^{-1}\left(b_m, N\right), N\right) - b\right] G'_{m_n|b}\left(b_m \,|x_n, N\right) db_m$$

- The first order condition is:

$$v_n\left(x_n, x_n, N\right) = b_n + \frac{G_{m_n|b}\left(b_n \,|x_n, N\right)}{G'_{m_n|b}\left(b_n \,|x_n, N\right)}$$

# Identification in FPSB Auctions with Private Values

## When all the bids are observed

- Assume $x_n = v_n$. From the first order condition:

$$x_n = b_n + \frac{G_{m_n}(b_n | x_n, N)}{G'_{m_n}(b_n | x_n, N)}$$

- Recall from its definition that $G_{m_n}(b_n | x_n, N)$ is the probability that $n$ wins the auction with $b_n$:

$$G_{m_n}(b_n | x_n, N) = \Pr\left[\max_{n' \in N \setminus n} \{b_{n'}\} \leq b_n | x_n, N\right]$$

- Thus if all the bids are observed then $G_{m_n}(b_n | x_n, N)$ is identified.
- Hence $v_n$ is identified (for all bidders in each sampled auction).
- Therefore the probability distribution of $(v_1, \ldots, v_N)$ in this specialization is identified for any correlation structure.

# Identification Fails in Common Value FPSB Auctions

## When all the bids are observed

- Recall that we defined:

$$v_n\left(x_n, x_{n'}, N\right) = E\left[v_n \middle| x_n \text{ and } \max_{n' \in N \backslash n}\left\{b_{n'}\right\} = \beta_n\left(x_{n'}, N\right)\right]$$

  and derived:

$$v_n\left(x_n, x_n, N\right) = b_n + \frac{G_{m_n}\left(b_n \left| x_n, N\right.\right)}{G'_{m_n}\left(b_n \left| x_n, N\right.\right)}$$

- The basic problem is that conditional on $N$ the RHS gives a number for each $n$, but the LHS is not a primitive of the model.
- Note that every common value model is observationally equivalent to a private value model found by setting $v_n = v_n\left(x_n, x'_n, N\right)$.
- Thus two common value models with possibly different $v_n\left(x_n, x_{n'}, N\right)$ but the same $v_n\left(x_n, x_n, N\right)$ are (also) observationally equivalent.