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45-812 Econometrics II

Mini 2 2024

ASSIGNMENT 3 (On Linear Models and Probabilty and Convergence)

There are eight equally weighted questions.

Question 1: What is $E\left[\beta_{OLS}^{(N)} - \beta_0\right]$ when $E\left[x_n\epsilon_n\right] \neq 0$?

Question 2: Suppose $\epsilon_n = \rho\epsilon_{n-1} + \upsilon_n$ for $n \in \{2, 3, \ldots\}$, with υ_n independently and identically distributed as a normal random variate with mean 0 and variance σ^2. and ϵ_1 similarly generated from an infinite past number of draws on $\upsilon_1, \upsilon_0, \upsilon_{-1}, \ldots$:

1. What is $var\left(\beta_{OLS}^{(N)}\right)$, the variance of $\beta_{OLS}^{(N)}$, in this case?
2. What is $var\left(\beta_{GLS}^{(N)}\right)$ for the model

Question 3: Let $x^{(N)} = (x_1, \ldots, x_N)$. Show $var\left(\beta_{IV}^{(N)}\right) \geq var\left(\beta_{OLS}^{(N)}\right)$ when $E\left[\epsilon_n \left| z_n\right.\right] = E\left[\epsilon_n \left| x_n\right.\right] = 0$ and

- $E\left[\epsilon_n\epsilon_m \left| x^{(N)}\right.\right] = \sigma^2$ if $m = n$
- but $E\left[\epsilon_n\epsilon_m \left| x^{(N)}\right.\right] = 0$ if $m \neq n$.

Question 4: Turning to constrained least squares:

1. Give an expression for $E\left[\beta_{CLS}^{(N)}\right]$?
2. What is $var\left(\beta_{CLS}^{(N)}\right)$ when $E\left[\beta_{CLS}^{(N)}\right] = \beta_0$?
3. Show $var\left(\beta_{CLS}^{(N)}\right) \leq var\left(\beta_{OLS}^{(N)}\right)$ when $Q\beta_0 = c$.
4. Show the bias of $\beta_{CLS}^{(N)}$ when $Q\beta_0 = c^* \neq c$.

Question 5: Suppose there are N observations on $(y_{1n}, y_{2n}, x_{1n}, x_{2n})$:

$$\begin{aligned} y_{1n} &= x_{1n}\beta_1 + \epsilon_{1n} \\ y_{2n} &= x_{2n}\beta_2 + x_{1n}\beta_1 + \epsilon_{2n} \end{aligned}$$

and assume $\epsilon_n \equiv (\epsilon_{1n}, \epsilon_{2n})$ is distributed bivariate normal and independent across n, and in addition independent of $x_n \equiv (x_{1n}, x_{2n})$, a $k \times 1$ vector, where x_{jn} is $k_j \times 1$ for $j \in \{1, 2\}$.

1. What is the mean and variance of $\beta_{2,S}^{(N)}$, the sequential estimator obtained from first regressing y_{1n} on x_{1n} to obtain the least squares estimator $\beta_{1,OLS}^{(N)}$ and then running the regression of $y_{2n} - x_{1n}\beta_{1,S}^{(N)}$ on x_{2n}.

2. Consider the distributional properties of $\beta_{2,OLS}^{(N)}$ (obtained by regressing y_{2n} on x_{1n} and x_{2n}), $\beta_{2,S}^{(N)}$ (the sequential estimator defined above), and $\beta_{GLS}^{(N)}$ (the estimator with the lowest covariance matrix).

Question 6: Prove the following

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
2. $\mathbb{P}(\phi) = 0$.
3. If $A \subseteq B$ then $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$.
4. If $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
5. If $A \subseteq B$ and $\mathbb{P}(B) = 0$ then $\mathbb{P}(A) = 0$.
6. If $A \subseteq B$ and $\mathbb{P}(A) = 1$ then $\mathbb{P}(B) = 1$.
7. $\mathbb{P}\left(\bigcup_{n \in N} A_n\right) \leq \sum_{n \in N} \mathbb{P}(A_n)$.
8. If $\mathbb{P}(A_n) = 0$ for all $n \in N$ then $\mathbb{P}\left(\bigcup_{n \in N} A_n\right) = 0$.
9. $\mathbb{P}\left(\bigcap_{n \in N} A_n\right) \geq 1 - \sum_{n \in N} [1 - \mathbb{P}(A_n)]$.
10. If $\{A_n\}_{n \in N}$ *partitions* Ω then $\mathbb{P}(B) = \sum_{n \in N} \mathbb{P}(A_n \bigcap B)$.

11. If $\mathcal{F}$ is a σ-algebra on Ω, show that $\phi \in \mathcal{F}$, and that $\mathcal{F}$ is closed under under countable intersections:

$$A_n \in \mathcal{F} \text{ for all } n \in N \Longrightarrow \bigcap_{n \in N} A_n \in \mathcal{F}$$

Question 7. Show $x_N(\omega) \stackrel{a.s.}{\to} 0$ iff $\delta_N \to 0$ where $\delta_1, \delta_2, \ldots$ is a sequence of real numbers and $x_N(\omega)$ is defined

$$x_N(\omega) = \begin{cases} 1 & \text{if } \omega \leq \delta_N \\ 0 & \text{if } \omega > \delta_N. \end{cases}$$

Question 8. Let $\varphi(x)$ denote a positive function, increasing on $(0, \infty)$, and symmetric about 0, meaning $\varphi(x) = \varphi(-x)$ and suppose x has probability density function $f(x)$. Provide a graphical demonstration of Chebychev's inequality:

- Use the horizontal axis to graph x
- On the vertical axis (use the upper half plane to) plot $\varphi(x)$.
- For some $u \in (0, \infty)$, plot $1\{|x| \geq u\} \varphi(x)$ and also $1\{|x| \geq u\} \varphi(u)$.
- Use the lower half plane to plot $f(x)$.
- Now plot $\varphi(x) f(x)$ and $1\{|x| \geq u\} \varphi(u) f(x)$
- Compare the integrals under the horizontal axis to establish and illustrate the inequality.