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45-812 Econometrics II

Mini 2

ASSIGNMENT 4

Question 1. (3 points) In the lecture slides we provided some intuition for a CLT for the iid case when all the moments exist (and fully characterize the distribution). Noting $N^{-1/2}S_N$ has mean 0 and variance σ^2, we considered the fourth moment:

$$\mathrm{E}\left(N^{-2}S_N^4\right) = N^{-2}\sum_{r=1}^{N}\sum_{s=1}^{N}\sum_{t=1}^{N}\sum_{u=1}^{N}\mathrm{E}\left(X_r X_s X_t X_u\right)$$

Since X_n are independently distributed, $E\left(X_r X_s X_t X_u\right) \neq 0$ only when:

$$r = s = t = u, \quad r = s \neq t = u, \quad r = t \neq s = u, \quad r = u \neq t = s$$

$$\begin{aligned} \Longrightarrow \mathrm{E}\left[N^{-2}S_N^4\right] &= N^{-2}\left\{N\mathrm{E}\left[X_n^4\right] + 3N\left(N-1\right)\sigma^4\right\} \\ &= o\left(N\right) + 3\left(N-1\right)N^{-1}\sigma^4 \end{aligned}$$

Thus the value of the fourth moment only depends on σ^2. Continuing the intuition for a simple CLT we claimed

1. A similar argument applies to all the even moments.

2. All the odd moments are asymptotically negligible.

Consequently, given a value for σ^2, the asymptotic distribution of $N^{-1/2}S_N$ does not depend on the distribution of X_n. In particular, when X_n is standard normal $\mathcal{N}\left(0, \sigma^2\right)$, so is $N^{-1/2}S_N \sim \mathcal{N}\left(0, \sigma^2\right)$ for all N. Using an induction argument prove 1 and 2 for the case in which only a finite number of moments characterize the distribution.

Question 2. (3 points) In the lecture notes we considered four test statistics for the null hypothesis that $g(\mu_0) = 0$ against the alternative that it is not:

1. Wald:

$$W_N = Ng(\mu_{un})' \left(\frac{\partial g(\mu_{un})}{\partial \mu} Q_N(\mu_{un})^{-1} \frac{\partial g(\mu_{un})'}{\partial \mu} \right)^{-1} g(\mu_{un})$$

2. J-statistic:

$$\begin{aligned} t_N &= N\left\{\left[f_N(\mu_{un})' S_N^{-1} f_N(\mu_{un})\right] - \left[f_N(\mu_r)' S_N^{-1} f_N(\mu_r)\right]\right\} \\ &= J_N(\mu_{un}) - J_N(\mu_r) \end{aligned}$$

3. Lagrange multiplier (LM) (or 'gradient test' or 'efficient score'):

$$L_N = N\left[f_N(\mu_r)' S_N^{-1} \frac{\partial f_N(\mu_r)}{\partial \mu}\right] Q_N^{-1} \left[\frac{\partial f_N(\mu_r)'}{\partial \mu} S_N^{-1} f_N(\mu_r)\right]$$

4. Minimum chi-squared (MC):

$$c_N = N(\mu_{un} - \mu_r^*)' Q_N(\mu_{un})^{-1} (\mu_{un} - \mu_r^*)$$

We then claimed without proof that:

1. If f is linear in μ then $t_N = L_N = c_N$.

2. If f and g are both linear in μ then: $w_N = t_N = L_N = c_N$.

Show what the expressions f g, Q, and S, defined in the lecture slides reduce to in these cases, and hence prove 1. and 2.

Question 3. (3 points) Let $\{g_N\}_{N=1}^{\infty}$ denote a sequence of positive real numbers. Using the definition of convergence in probability (in terms of exception sets), prove:

$$\begin{aligned} o_p(g_N) + o_p(g_N) &= o_p(g_N) \\ o_p(g_N) O_p(g_N) &= o_p(g_N) \end{aligned}$$

Now consider two sequences of positive real numbers, denoted by $\{g_{1N}\}_{N=1}^{\infty}$ and $\{g_{2N}\}_{N=1}^{\infty}$. What can we say about:

$$o_p(g_{1N}) + o_p(g_{2N})$$

$$o_p(g_{1N}) O_p(g_{2N})$$

Question 4. (6 points) Suppose (Y_n, X_n, Z_n) is *iid* for $n \in \{1, 2, \ldots\}$ where takes values on the real line, $X_n \equiv (X_{1n}, X_{2n})$ and $Z_n = (X_{1n}, Z_{1n}, Z_{2n})$. Assume that the only additional thing that our model says is:

$$E\left[Y_n - \beta_0^* - \beta_1^* \ln X_{1n} - \beta_2^* X_{2n}^{\beta_3^*} | Z_n\right] = 0$$

for some $(\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*) \in \mathbb{R}^4$:

1. Write down sufficient conditions for the model to obtain a consistent estimator that converges at rate $\sqrt{N}$ to an asymptotically normal random variable centered at $(\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*)$.

2. Obtain a consistent estimator for $(\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*)$.

3. Given the instruments you picked for the previous question, what is the optimal weighting matrix for your specification?

4. What are the optimal instruments for this model?

5. Explain in precise detail for this particular model, all the necessary steps to obtain the optimal instrumental variables estimator.

6. For the specialization defined by assuming that $X_{2n} \equiv Z_{2n}$, prove the optimal instrumental variables estimator is asymptotically efficient.