

# Probability and Convergence

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# Probability Measure

## Collections of sets and sigma algebras

- Let  $\Omega$  denote a collection of events and  $\phi \subseteq \Omega$  the empty set.
- Let  $N \equiv \{1, 2, \dots\}$  denote the counting numbers.
- Define  $A^c \subseteq \Omega$ , the *complement* of  $A \subseteq \Omega$  such that:

$$A \cap A^c = \phi \text{ and } A \cup A^c = \Omega$$

- Let  $B \setminus A$  denote the *set difference* of subtracting  $A$  from  $B$ :

$$B \setminus A \equiv (A \cap B)^c \cap B$$

- A countable collection of events  $\{A_n\}_{n \in N}$  *partitions*  $\Omega$  iff:

$$\bigcap_{n \in N} A_n = \phi \text{ and } \bigcup_{n \in N} A_n = \Omega$$

- A  $\sigma$ -algebra on  $\Omega$ , denoted by  $\mathcal{F}$ , is a collection of subsets containing  $\Omega$ , and closed under:

- complements:  $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
- countable unions:  $A_n \in \mathcal{F}$  for all  $n \in N \implies \bigcup_{n \in N} A_n \in \mathcal{F}$

# Probability Measure

## Kolmogorov Axioms

- A *probability measure*  $\mathbb{P}$  is a real valued function, defined on  $A \subseteq \Omega$  satisfying the following axioms:
  - 1  $\mathbb{P}(A) \geq 0$  for every event  $A \subseteq \Omega$ .
  - 2  $\mathbb{P}(\Omega) = 1$
  - 3 If  $A_m \cap A_n = \emptyset$  for all  $n \neq m$  then  $\mathbb{P}(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mathbb{P}(A_n)$ .
- The first two axioms essentially bound how events are weighed.
- In this way we avoid the conundrum of adding infinities to each other or subtracting infinity from infinity.
- Aside from that, the choices of "0" and "1" in the first two axioms is for convenience.
- The third axiom, known as *countable additivity*, helps define sums.
- Note the interval  $[0, 1]$  has length one but is the union of an uncountable of points  $r \in [0, 1]$ , each point having length zero.

# Probability Measure

## Properties of measures

- 1  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ .
- 2  $\mathbb{P}(\emptyset) = 0$ .
- 3 If  $A \subseteq B$  then  $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$ .
- 4 If  $A \subseteq B$  then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .
- 5 If  $A \subseteq B$  and  $\mathbb{P}(B) = 0$  then  $\mathbb{P}(A) = 0$ .
- 6 If  $A \subseteq B$  and  $\mathbb{P}(A) = 1$  then  $\mathbb{P}(B) = 1$ .
- 7  $\mathbb{P}(\bigcup_{n \in N} A_n) \leq \sum_{n \in N} \mathbb{P}(A_n)$ .
- 8 If  $\mathbb{P}(A_n) = 0$  for all  $n \in N$  then  $\mathbb{P}(\bigcup_{n \in N} A_n) = 0$ .
- 9  $\mathbb{P}(\bigcap_{n \in N} A_n) \geq 1 - \sum_{n \in N} [1 - \mathbb{P}(A_n)]$ .
- 10 If  $\{A_n\}_{n \in N}$  *partitions*  $\Omega$  then  $\mathbb{P}(B) = \sum_{n \in N} \mathbb{P}(A_n \cap B)$ .

# Probability Measure

## Conditional probability and independence

- Given  $(\Omega, \mathcal{F}, P)$  suppose  $B \in \mathcal{F}$  and  $P(B) \neq 0$ .
- Then for all  $A \in \mathcal{F}$  we define the probability of  $A$  conditional on  $B$  as:

$$P(A|B) \equiv P(A \cap B) / P(B)$$

- More generally:

$$P\left(\bigcap_{n \in N} A_n\right) \equiv P(A_1) \prod_{n=2}^N P\left(A_n \mid \bigcap_{k=1}^{n-1} A_k\right)$$

- We say  $A$  and  $B$  are independent events iff:

$$P(A \cap B) = P(A) P(B) \text{ or equivalently } P(A|B) = P(A)$$

or more generally:

$$P\left(\bigcap_{n \in N} A_n\right) = \prod_{n \in N} P(A_n)$$

# Probability Measure

## Random variables

- Denote the elements of  $\Omega$  by  $\omega \in \Omega$ .
- Note, depending on how  $\mathcal{F}$  is defined, perhaps  $\omega \notin \mathcal{F}$ .
- Real valued, measurable functions of  $\Omega$  are called random variables.
- That is  $x(\omega) : \Omega \rightarrow \mathbb{R}$ , and for all Borel sets  $B \subseteq \mathbb{R}$ :

$$X^{-1}(B) \equiv \{\omega : x(\omega) \in B\} \in \mathcal{F}$$

- We write the conditional expectation of  $x(\omega)$  as:

$$E(x|B) = \int_B x(\omega) dP(A|B)$$

- If  $x(\omega) = (x_1(\omega), x_2(\omega), \dots) \in \mathbb{R}^\infty$  is measurable with respect to  $\mathcal{B}_\infty$ , then:

$$\begin{aligned} P(x(\omega)) &= P\left(\bigcap_{n=1}^{\infty} x_n(\omega)\right) \\ &= P(x_1(\omega)) \prod_{n=2}^{\infty} P(x_n(\omega) | x_1(\omega), \dots, x_{n-1}(\omega)) \end{aligned}$$

# Probability Measure

## Independence and identically distributed random variables

- We say  $x(\omega)$  is independent iff:

$$P(x(\omega)) = \prod_{n=1}^{\infty} P(x_n(\omega))$$

- Supposing  $x(\omega)$  is independent, then it is identically distributed as well if:

$$P(x_n(\omega)) = P(x_m(\omega))$$

for all  $n, m, \omega$ .

# Statistical Inference

## Model

- For each  $N = 1, 2, \dots$ , let  $(\Omega, \mathcal{F}_N, P)$  denote a probability space, where we now interpret:
  - $\omega \in \Omega$ : a world history, or an ordering of the population, from beginning to end.
  - $N$ : sample size showing how far history (many individuals) has progressed (have been interviewed).
  - $\mathcal{F}_N$ : set of all possible (partial) histories showing the distinguishing features of  $\omega$  that can be observed from sample of size  $N$ . By construction  $\mathcal{F}_N \subseteq \mathcal{F}_{N+1} \subseteq \dots$
  - $P$ : probability measure over possible (world) histories.
  - $\theta \in \Theta$ : unknown parameters of interest defining  $P$ .
  - $h_N(\omega; \theta) : \Omega \times \Theta \rightarrow \mathbb{R}^q$ : a function of the sample, measurable with respect to  $(\Omega, \mathcal{F}_N, P)$ .



# Statistical Inference

## Motivation

- Consider a statistic:

$$\theta \left( x^{(N)} \right) \equiv \theta_N$$

- On basis of this statistic, suppose we want to estimate  $\theta_0$ , the parameters defining the data generating process.
- Ideally this involves deriving:

$$\Pr \{ \theta_N \leq \theta \mid \theta_0 \} \quad \forall \theta$$

- We often focus on the first two moments:

$$E [ \theta_N \mid \theta_0 ] \text{ and } \text{var} ( \theta_N \mid \theta_0 )$$

- This approach is usually intractable for nonlinear models.

# Statistical Inference

## Three questions

- Rather than ask what is the probability distribution of  $\theta_N$  given  $\theta_0$  we ask some less demanding questions about the limiting properties of  $\theta_N$  as  $N \rightarrow \infty$ .
- **First question:** For some definition of " $\rightarrow$ ", is there a sense in which:

$$\theta_N \text{ "} \rightarrow \text{" } \theta_0?$$

- **Second question:** Suppose the answer to the first question is "yes". Is there a sense in which how quickly  $\theta_N \text{ "} \rightarrow \text{" } \theta_0$ ? A more precise way of stating this question is to ask for what values of  $\alpha > 0$  does  $N^\alpha (\theta_N - \theta_0) \text{ "} \rightarrow \text{" } 0$ ?
- **Third question:** Define  $\mu_N^\alpha \equiv N^\alpha (\theta_N - \theta_0)$ , and let  $G_N \left( \cdot \mid \{x_n\}_{n=1}^N \right)$  denote the probability distribution function of  $\mu_N^\alpha$  given  $\theta_0$  and conditional on the sequence  $\{x_n\}_{n=1}^N$ . Is there a sense in which  $G_N \left( \cdot \mid \{x_n\}_{n=1}^N \right) \text{ "} \rightarrow \text{" } G \left( \cdot \mid \{x_n\}_{n=1}^\infty \right)$ ?

# Convergence

## Exception sets

- For  $(\Omega, \mathcal{F}, P)$ , a probability space, and  $\omega \in \Omega$ , a history, define an exception set as:

$$A_{Nr} = \left\{ \omega : |\theta_N(\omega) - \theta(\omega)| > \frac{1}{r} \right\}$$

- Thus  $A_{Nr}$  depicts for which histories it is true that our estimator  $\theta_N(\omega)$  differs from  $\theta(\omega)$  by more than  $\frac{1}{r}$ .
- Note that  $A_{Nr} \subseteq A_{N,r+1} \subseteq \dots$
- Convergence is based on the notion that  $A_{Nr}$  shrinks as  $N$  increases.

# Convergence

## Convergence in probability

- Convergence in probability means the probability of an exception set occurring converges to zero.
- We denote convergence in probability by:

$$x_N \xrightarrow{P} x \quad (\text{or } \text{plim } x_N = x)$$

- Formally  $x_N \xrightarrow{P} x$  if and only if for all  $r \in \{1, 2, \dots\}$ :

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} \Pr(A_{Nr}) \\ &= \lim_{N \rightarrow \infty} \Pr \left\{ \omega : |x_N(\omega) - x(\omega)| > \frac{1}{r} \right\} \quad \forall r = 1, 2, \dots \end{aligned}$$

- That is for each  $(r, \delta) \in \{1, 2, \dots\} \times \mathbb{R}$  there exists some  $N^*$  such that for all  $N > N^*$ :

$$\Pr \left\{ \omega : |x_N(\omega) - x(\omega)| > \frac{1}{r} \right\} \equiv \Pr(A_{Nr}) < \delta$$

# Convergence

An example of convergence in probability

- Let:

$$\Omega = [0, 1],$$

$$\mathcal{F} \equiv \mathcal{B}[0, 1] \text{ for the Borel sets, and}$$

$$\mathcal{L} \equiv \Pr\{\omega \in [0, a]\} = a \quad \text{on } [0, 1]$$

- Let  $\delta_1, \delta_2, \dots$  be a sequence of real numbers and:

$$x_N(\omega) \equiv \begin{cases} 1 & \text{if } \omega \leq \delta_N \\ 0 & \text{if } \omega > \delta_N. \end{cases}$$

- Then

$$x_N(\omega) \xrightarrow{p} 0 \quad \text{iff } \delta_N \rightarrow 0.$$

# Convergence

Extending convergence in probability to vectors of random variables

- Now suppose  $x_N(\omega)$  and  $x(\omega)$  are  $K$  dimensional vector of random variables expressed as:

$$\begin{aligned}x_N(\omega) &\equiv (x_{1N}(\omega), \dots, x_{KN}(\omega)) \\x(\omega) &\equiv (x_1(\omega), \dots, x_K(\omega))\end{aligned}$$

- We say  $x_N(\omega) \xrightarrow{P} x(\omega)$  iff::

$$\begin{aligned}&\lim_{N \rightarrow \infty} \Pr \left\{ \omega : \|x_N(\omega) - x(\omega)\| > \frac{1}{r} \right\} \\&\equiv \lim_{N \rightarrow \infty} \Pr \left\{ \omega : \left\| \sum_{k=1}^K [x_{kN}(\omega) - x_k(\omega)]^2 \right\| > \frac{1}{r} \right\} \\&= 0\end{aligned}$$

- Note that  $x_N(\omega) \xrightarrow{P} x(\omega)$  iff  $x_{kN}(\omega) \xrightarrow{P} x_k(\omega)$  for all  $k$ .

# Convergence

Almost sure convergence defined

- Consider subsets  $A_N \subset \Omega$  for  $N = 1, 2, \dots$  and define:

$$\limsup_N A_N \equiv \bigcap_{N=1}^{\infty} \bigcup_{K=N}^{\infty} A_K \quad \text{and} \quad \liminf_N A_N \equiv \bigcup_{N=1}^{\infty} \bigcap_{K=N}^{\infty} A_K$$

- This leads us to focus on:

$$A_r^* \equiv \limsup_N A_{Nr} = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \left\{ \omega : |x_k(\omega) - x(\omega)| > \frac{1}{r} \right\}$$

- We say  $x_N \xrightarrow{a.s.} x$  (or  $x_N$  converges to  $x$  with probability 1) if and only if:

$$0 = \Pr \left( \bigcup_{r=1}^{\infty} A_r^* \right) = \Pr \left( \bigcup_{r=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \left\{ \omega : |x_k(\omega) - x(\omega)| > \frac{1}{r} \right\} \right)$$

# Convergence

## $L_p$ convergence

- We say  $x_N$  converges to  $x$  in  $L_p$  for  $p = \{1, 2, \dots\}$  or  $x_N \xrightarrow{L_p} x$  if and only if:

$$\lim_{N \rightarrow \infty} E[|x_N - x|^p] = 0$$

- If  $p = 2$  we say “convergence in quadratic mean” or “convergence in mean square”.



# How are these Convergence Concepts Related?

## Chebychev's inequality

- Chebychev's inequality is a useful tool for understanding how the relationship between these convergence concepts
- It states: If  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is positive, increasing on  $(0, \infty)$ , symmetric ( $\varphi(u) = \varphi(-u)$ ), and  $x$  is a random variable on  $(\Omega, \mathcal{F}, P)$ , then for all  $u > 0$ :

$$\varphi(u) P(|x| \geq u) \leq E(\varphi(x))$$

- To prove Chebychev's inequality, note that:

$$\begin{aligned} E(\varphi(x)) &= \int_{\Omega} \varphi(x(\omega)) dP \\ &\geq \int_{\Omega} \mathbf{1}_{\{|x| \geq u\}}(\omega) \varphi(x(\omega)) dP \\ &\geq \int_{\Omega} \mathbf{1}_{\{|x| \geq u\}}(\omega) \varphi(u) dP \\ &= \varphi(u) P(|x| \geq u) \end{aligned}$$

# How are these Convergence Concepts Related?

Two applications of Chebychev's inequality

- To illustrate Chebychev's inequality set:

$$\varphi(x) = x^2 \quad \text{and} \quad u = \sigma$$

to obtain

$$\Pr(|x| \geq \sigma) \leq \frac{E(x^2)}{\sigma^2}$$

## Lemma

*If  $x_N \xrightarrow{L_p} x$  then  $x_N \xrightarrow{P} x$ .*

# How are these Convergence Concepts Related?

## Proof.

For  $p > 0$  set:

$$\varphi(u) = |u|^p \text{ and } u = \frac{1}{r}$$

Since  $\varphi(u)$  is symmetric and increasing, we can apply Chebychev's inequality to obtain:

$$\Pr \left( |x_N - x| \geq \frac{1}{r} \right) \leq \mathbb{E} [|x_N - x|^p] r^p$$

By hypothesis:

$$\lim_{N \rightarrow \infty} \mathbb{E} [|x_N - x|^p] = 0$$

so:

$$\lim_N \Pr \left( |x_N - x| \geq \frac{1}{r} \right) = 0$$



# How are these Convergence Concepts Related?

## Lemma

$x_N \xrightarrow{a.s.} x$  does not imply  $x_N \xrightarrow{L_p} x$ .

# How are these Convergence Concepts Related?

## Proof.

Let  $(\Omega, \mathcal{F}, P) \equiv \{[0, 1], \mathcal{B}[0, 1], \mathcal{L}\}$  and define:

$$x_N(\omega) = \begin{cases} 2^N & \text{if } \omega \in (0, 1/N) \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{Then } \Pr(\bigcup_{r=1}^{\infty} A_r^*) &= \Pr(\bigcup_{r=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\omega : x_k(\omega) > 1/r\}) \\ &\leq \Pr(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\omega : x_k(\omega) \neq 0\}) \\ &= \Pr(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\omega \in (0, 1/N)\}) \\ &= \Pr(\bigcap_{N=1}^{\infty} \{\omega \in (0, 1/N)\}) \\ &= 0 \end{aligned}$$

Therefore  $x_N \xrightarrow{a.s.} 0$ . But  $E[|x_N|^p] = (2^N/N)^p \rightarrow \infty$  so  $x_N \not\xrightarrow{L_p} 0$ .



# How are these Convergence Concepts Related?

Almost sure convergence implies convergence in probability

## Theorem

(Theorem 18.3 J. Davidson, *Stochastic limit theory*, p. 283):  $x_N \xrightarrow{a.s.} x$  if and only if for all  $\varepsilon > 0$ :

$$\lim_{m \rightarrow \infty} \Pr \left[ \sup_{N \geq m} |x_N - x| \leq \varepsilon \right] = 1$$

## Lemma

$$x_N \xrightarrow{a.s.} x \implies x_N \xrightarrow{p} x$$

# How are these Convergence Concepts Related?

Proof of lemma

Proof.

$$\begin{aligned} & \sup_{N \geq m} |x_N - x| \geq |x_m - x| \\ \Rightarrow & \Pr \left[ \sup_{N \geq m} |x_N - x| \leq \varepsilon \right] \leq \Pr [ |x_m - x| \leq \varepsilon ] \\ \Rightarrow & \lim_{m \rightarrow \infty} \Pr \left[ \sup_{N \geq m} |x_N - x| \leq \varepsilon \right] = 1 \leq \lim_{m \rightarrow \infty} \Pr [ |x_m - x| \leq \varepsilon ] \\ \Rightarrow & \lim_{m \rightarrow \infty} \Pr [ |x_m - x| \leq \varepsilon ] = 1 \end{aligned}$$

But:

$$\begin{aligned} & \lim_{m \rightarrow \infty} \Pr [ |x_m - x| \leq \varepsilon ] = 1 \\ \Rightarrow & \lim_{m \rightarrow \infty} \Pr [ |x_m - x| > \varepsilon ] = 0. \end{aligned}$$

# How are these Convergence Concepts Related?

## Lemma

$$x_N \xrightarrow{L_p} x \text{ does not imply } x_N \xrightarrow{\text{a.s.}} x$$

Proof: Let  $(\Omega, \mathcal{F}, P) \equiv \{[0, 1], \mathcal{B}[0, 1], \mathcal{L}\}$  and define  $x_N(\omega)$  as:

$$x_1(\omega) = 1_{[0, 1/2]}(\omega), \quad x_2(\omega) = 1_{[1/2, 1]}(\omega)$$

$$x_3(\omega) = 1_{[0, 1/3]}(\omega), \quad x_4(\omega) = 1_{[1/3, 2/3]}(\omega), \quad x_5(\omega) = 1_{[2/3, 1]}(\omega), \quad \dots$$

Relative to  $x(\omega) = 0$ , for all  $p$ :

$$E(|x_N|^p) = \int_{\Omega} 1_{A_N(\omega)} dP = \frac{1}{k+1} \text{ if } \frac{k(k+1)}{2} \leq N \leq \frac{(k+1)(k+2)}{2}$$

Therefore  $x_N \xrightarrow{L_p} 0$  for all  $p$  because  $(k+1)^{-1} \rightarrow 0$  as  $N \rightarrow \infty$ .

But  $\forall \omega \in \Omega$  and  $N \in \{1, 2, \dots\}$  there exists  $k$  such that  $x_k(\omega) = 1$ :

$$\bigcup_{r=1}^{\infty} A_r^* = \bigcup_{r=1}^{\infty} \lim_{N \rightarrow \infty} \bigcup_{k=N}^{\infty} \left\{ \omega : x_k(\omega) > \frac{1}{r} \right\} = \bigcup_{r=1}^{\infty} \{ \omega : \omega \in \Omega \} = \Omega$$



# How are these Convergence Concepts Related?

## Lemma

Suppose

$$|x_N(\omega)| \leq y(\omega) \in L^p$$

If  $x_N(\omega) \xrightarrow{P} 0$ , then  $x_N(\omega) \xrightarrow{L^p} 0$ .

## Proof.

See Chung (1974, page 67): *A Course in Probability Theory*

# Order of Magnitude

## Relative magnitude of real sequences

- Let  $\{y_N\}_{n=1}^{\infty}$  denote a sequence of real numbers, and  $\{g_N\}_{n=1}^{\infty}$  a sequence of positive real numbers.
- We say  $y_N$  is at most of order  $g_N$ , and write  $y_N = O(g_N)$  if there exists a real number  $M$  such that for all  $N$ :

$$\frac{|y_N|}{g_N} \leq M$$

- We say  $y_N$  is of smaller order than  $g_N$  and write  $y_N = o(g_N)$  if and only if:

$$\frac{y_N}{g_N} \rightarrow 0.$$

That is for any  $\varepsilon > 0$  there exists an  $N_\varepsilon$  such that  $\frac{|y_N|}{g_N} < \varepsilon$  for all  $N \in \{N_\varepsilon + 1, N_\varepsilon + 2, \dots\}$ .

# Order of Magnitude

## Order in probability

- Now let  $x_N$  be a random variable with respect to  $(\Omega, \mathcal{F}, P)$ .
- We say  $x_N$  is at most of order  $g_N$  writing:

$$x_N = O_p(g_N)$$

if there exists a positive number  $M_r$  for all  $r \in \{1, 2, \dots\}$  such that for all  $N$ :

$$\Pr \left[ \frac{|x_N|}{g_N} \geq M_r \right] \leq \frac{1}{r}$$

- We say that  $x_N$  is of smaller order than  $g_N$  in probability, writing:

$$x_N = o_p(g_N)$$

if:

$$\frac{x_N}{g_N} \xrightarrow{p} 0.$$

# Order in Magnitude

Second moments imply being bounded in probability

## Lemma

Let  $x_1, x_2, \dots$  be a sequence of random variables with respect to  $(\Omega, \mathcal{F}, P)$ , and  $g_1, g_2, \dots$  a sequence of positive real numbers such that

$$E(x_N^2) = O(g_N^2)$$

Then

$$x_N = O_p(g_N)$$

- For example suppose  $x_N \sim \mathcal{N}(0, \sigma^2)$  that for all  $N$ . Then:

$$E(x_N^2) = \sigma^2 = O(1)$$

and by the lemma  $x_N = O_p(1)$ .

# Order in Magnitude

## Proving the connection

### Proof.

By the premise there exists  $M < \infty$ , such that:

$$E [x_N^2 / g_N^2] \leq M$$

Also by Chebychev's inequality, for all  $M_r > 0$ :

$$\Pr \left[ \frac{|x_N|}{g_N} \geq M_r \right] \leq \frac{E [x_N^2 / g_N^2]}{M_r^2}$$

For any  $r$  choose  $M_r$  to satisfy the inequality:

$$M_r > (Mr)^{1/2} \Rightarrow M_r^2 > Mr \Rightarrow \frac{M}{M_r^2} \leq \frac{1}{r}$$

$$\text{Then } \Pr \left[ \frac{|x_N|}{g_N} \geq M_r \right] \leq \frac{E [x_N^2 / g_N^2]}{M_r^2} \leq \frac{M}{M_r^2} \leq \frac{1}{r}.$$

- ① If  $\mathcal{F}$  is a  $\sigma$ -algebra on  $\Omega$ , show that  $\phi \in \mathcal{F}$ , and that  $\mathcal{F}$  is closed under under countable intersections:

$$A_n \in \mathcal{F} \text{ for all } n \in N \implies \bigcap_{n \in N} A_n \in \mathcal{F}$$

- ② Show  $x_N(\omega) \xrightarrow{a.s.} 0$  iff  $\delta_N \rightarrow 0$  where  $\delta_1, \delta_2, \dots$  is a sequence of real numbers and  $x_N(\omega)$  is defined

$$x_N(\omega) = \begin{cases} 1 & \text{if } \omega \leq \delta_N \\ 0 & \text{if } \omega > \delta_N. \end{cases}$$