# Continuous Choices

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Structural Econometrics

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- Competitive equilibrium is the bedrock of economics:
  - Consumers reveal their preferences through their choices (*three axioms supporting consumer choice theory*);
  - Given the price of each commodity, consumers and producers buy or sell as many units as they wish (*individual optimization*);
  - At those prices the market for each commodity clears, supply matching demand (*existence of equilibrium*);
  - All potential gains from trade are realized, the economy achieving a Pareto efficient allocation (*welfare theorems*).

# Competitive Equilibrium

## The budget constraint

- Consider the following model of consumer choice in a static model.
- The budget constraint for household $n$ is:

$$\sum_{k=1}^{K} p_k c_{nk} \leq B_n \tag{1}$$

- The utility function for household $n$ is:

$$u\left(c_{n1}, \ldots, c_{nK}\right) \tag{2}$$

- Let:
  1. $\eta_n$ denote the Lagrange multiplier associated with $(4)$
  2. $p_k$ denote the spot price of $k$ at $t$ (conditional the state)
  3. the first good be a numeraire.
  4. $c_n \equiv \left(c_{n1}, \ldots, c_{nK}\right)$ denote the consumption vector of $n$.
- Household $n$ maximizes $(5)$ subject to $(4)$.
- Then the first order condition for an interior solution for $k$ is:

$$u_k\left(c_n\right) \equiv \frac{\partial u\left(c_n\right)}{\partial c_{nk}} = \eta_n p_k \tag{3}$$

# Competitive Equilibrium

## Marginal rates of substitution in equilibrium

- Temporarily dropping for convenience the subscript $n$, the individual identifier, and setting $p_1 \equiv 1$, there are:
  - $(K-1)$ equations corresponding to the spot markets:

$$MRS_k(c_n) \equiv \frac{u_k(c_n)}{u_1(c_n)} = p_k$$

- Given $\eta_n$, these $K-1$ marginal rates of substitution equations fully characterize an interior equilibrium consumption of $n$.

# Competitive Equilibrium

## Example

- For example suppose:

$$u(c_n) \equiv \sum_{k=1}^{K} \frac{\exp(x_n B_k + \epsilon_{nk})}{\alpha_k + 1} c_{nk}^{\alpha_k + 1}$$

where:

- $x_n$ is a vector of observed demographic variables that characterize household $n$ (such as family size, ages, education levels).
- $B_k$ is a coefficient vector to be estimated thta weight each of the demographic variables.
- $\epsilon_{nk}$ is an unobserved variable, which we assume is independently and identically distributed (*iid*).

# Competitive Equilibrium

## First order condition for the example

- Focusing on the first two goods we have:

$$\begin{aligned} p_2 &= MRS_2\left(c_n\right) \\ &= \exp\left[x_{nt}\left(B_2 - B_1\right) + \epsilon_{nt2} - \epsilon_{nt1}\right] \frac{c_{nt2}^{\alpha_2}}{c_{nt1}^{\alpha_1}} \end{aligned}$$

Taking logarithms:

$$\begin{aligned} & \epsilon_{nt2} - \epsilon_{nt1} \\ = \;& x_{nt}\left(B_1 - B_2\right) + \alpha_1 \ln\left(c_{nt1}\right) - \alpha_2 \ln\left(c_{nt2}\right) + \ln p_{t2} \end{aligned}$$

# Competitive Equilibrium

## Estimation

- For any instrument vector $z_{nt}$ satisfying:

$$E\left[\epsilon_{nt}\left|z_{nt}\right.\right]=0$$

we have:

$$E\left\{z_{nt}\left[x_{nt}\left(B_1-B_2\right)+\alpha_1\ln\left(c_{nt1}\right)-\alpha_2\ln\left(c_{nt2}\right)+\ln p_{t2}\right]\right\}=0$$

- A GMM estimator now comes from setting

$$0=A\sum_{n=1}^{N}z_{nt}\left[x_{nt}\left(B_1-B_2\right)+\alpha_1\ln\left(c_{nt1}\right)-\alpha_2\ln\left(c_{nt2}\right)+\ln p_{t2}\right]$$

- The usual large sample properties apply.

# Complete Markets

## A dynamic generalization (Altug and Miller, 1990)

- In the dynamic analogue to a static model of competitive equilibrium (Debreu, 1959 Chapter 7):
  - Commodities are indexed by time and the state of the world.
- Most econometric analysis also make several auxiliary assumptions.

1. The expected utility hypothesis holds.
2. Subjective beliefs match probability transitions.
3. Preferences are time additively separable up to a finite vector of human capital, habit persistence and nondurables.
4. Current utility payoffs are sufficiently smooth that individual agents only require a small number of securities to achieve the equilibrium allocations of the complete markets.

# Complete Markets

## Commodity prices

- Let $\{\mathcal{F}_t\}_{t=0}^{\infty}$ denote a sequence of $\sigma$-algebras with measure $\mathcal{P}$ that reflects how history unfolds as the economy evolves.
- Each period $t \in \{\underline{n}, \ldots, \overline{n}\}$ household $n$ consumes $(c_{nt1}, \ldots, c_{ntK})$.
- Define a commodity by the triplet $(k, t, A)$.
- Let $p_{tk}(A)$ denote the date zero price of receiving a unit of $k$ at $t$ in the event of $A \in \mathcal{F}_t$ occurring:

$$p_{tk}(A) = \int_A \lambda_{tk}(\omega) \mathcal{P}(d\omega)$$

- The Radon-Nikodym derivative $\lambda_{tk}(\omega)$ converts the probability of events into a commodity price measure.

# Complete Markets

## A lifetime budget constraint

- Assume markets are complete and there is a competitive market for every commodity defined on $k$, $t$, and $\{\mathcal{F}_t\}_{t=0}^{\infty}$ .
- The assumption of complete markets implies the consumer budget set is a single lifetime budget constraint, rather than a sequence of period-specific budget constraints.
- The lifetime budget constraint for $n$ is:

$$E_0 \left[ \sum_{t=\underline{n}}^{\overline{n}} \sum_{k=1}^{K} \lambda_{tk} c_{ntk} \right] \leq B_n \tag{4}$$

- Compared with the representative consumer model, we:
  - drop the representative consumer assumption;
  - assume a competitive equilibrium exhausts the gains from trade.

# Complete Markets

## Time additive preferences, maximization and the first order condition

- Suppose households obey the expected utility hypothesis, preferences taking the time additive form:

$$E_0 \left[ \sum_{t=\underline{n}}^{\bar{n}} \beta^{t-\underline{n}} u_{t-\underline{n}} \left( c_{nt1}, \ldots, c_{ntK} \right) \right] \tag{5}$$

- Let:
  1. $\eta_n$ denote the Lagrange multiplier associated with $(4)$
  2. $p_{tk}$ denote the spot price of $k$ at $t$ (conditional the state)
  3. the first good be a numeraire and define $\lambda_t \equiv \lambda_{t1}$.
  4. $t_n \equiv t - \underline{n}$ denote the age of the household
  5. $c_{nt} \equiv (c_{nt1}, \ldots, c_{ntK})$ denote the consumption vector of $n$ at $t$.
- Household $n$ maximizes $(5)$ subject to $(4)$.
- Then the first order condition for an interior solution for $k$ is:

$$\beta^{t_n} u_{t_n,k} \left( c_{nt} \right) \equiv \beta^{t_n} \frac{\partial u_{t_n} \left( c_{nt} \right)}{\partial c_{ntk}} = \eta_n \lambda_{tk} \equiv \eta_n \lambda_t p_{tk} \tag{6}$$

# Complete Markets

## Marginal rates of substitution in equilibrium

- Temporarily dropping for convenience the subscript $n$, the individual identifier, and setting $p_{t1} \equiv 1$, there are:
  1. $(K-1)(\overline{n}-\underline{n})$ equations corresponding to the spot markets:

$$MRS_{tk}(c_t) \equiv \frac{u_{tk}(c_t)}{u_{t1}(c_t)} = p_{tk}$$

  2. $(\overline{n}-\underline{n})-1$ equations pertaining to the numeraire that intertemporally balance consumption:

$$MRS_t(c_t, c_{t+1}) \equiv \frac{\beta u_{t+1,1}(c_{t+1})}{u_{t1}(c_t)} = \frac{\lambda_{t+1}}{\lambda_t}$$

- Given $\eta_n$, we can show that these $K(\overline{n}-\underline{n})-1$ marginal rates of substitution equations fully characterize an interior equilibrium consumption of $n$.

# Complete Markets

## The remaining marginal rates of substitution and their equilibrium conditions

- All remaining marginal rates of substitution functions are fully described by $MRS_{tk}\left(c_t\right)$ and $MRS_t\left(c_t, c_{t+1}\right)$ because:

$$MRS_{t,s,k,l}\left(c_t, c_s\right) \equiv \frac{\beta^{s-t} u_{sl}\left(c_{sl}\right)}{u_{tk}\left(c_{nt}\right)} = \frac{MRS_{sl}\left(c_s\right)}{MRS_{tk}\left(c_t\right)} \prod_{r=t}^{s} MRS_r\left(c_r, c_{r+1}\right)$$

- Likewise all the remaining contingent prices are described by the vector sequence of spot prices $\left\{\left(p_{t2}, \ldots, p_{tK}\right)\right\}_{t=\underline{n}}^{\overline{n}}$ along with the contingent price sequence for the numeraire $\left\{\lambda_t\right\}_{t=\underline{n}}^{\overline{n}}$, because:

$$\frac{p_{sl}}{p_{tk}} \prod_{r=t}^{s-1} \frac{\lambda_{r+1}}{\lambda_r} = \frac{\lambda_s p_{sl}}{\lambda_t p_{tk}}$$

- It follows that additional equalities of the form:

$$MRS_{t,s,k,l}\left(c_t, c_s\right) = \frac{\lambda_s p_{sl}}{\lambda_t p_{tk}}$$

provide neither additional restrictions nor additional parameters.

# Complete Markets

## Estimation using the equilibrium conditions for the spot markets

- For example suppose:

$$u_t\left(c_{nt}\right) \equiv \sum_{k=1}^{K} \frac{\exp\left(x_{nt} B_k + \epsilon_{ntk}\right)}{\alpha_k + 1} c_{ntk}^{\alpha_k + 1}$$

- The equilibrium conditions for the spot markets are identical to those used to estimate the static model.
- Appealing to the marginal rate of substitution condition:

$$\begin{aligned} p_{t2} &= MRS_{t2}\left(c_{nt}\right) \\ &= \exp\left[x_{nt}\left(B_2 - B_1\right) + \epsilon_{nt2} - \epsilon_{nt1}\right] \frac{c_{nt2}^{\alpha_2}}{c_{nt1}^{\alpha_1}} \end{aligned}$$

we could form a GMM estimator using instruments $z_{nt}$ and solve for the estimated coefficients from:

$$0 = A \sum_{n=1}^{N} z_{nt}\left[x_{nt}\left(B_1 - B_2\right) + \alpha_1 \ln\left(c_{nt1}\right) - \alpha_2 \ln\left(c_{nt2}\right) + \ln p_{t2}\right]$$

# Complete Markets

## Estimating the intertemporal rates of substitution

- Similarly:

$$\frac{\lambda_{t+1}}{\lambda_t} = \beta \exp\left[\left(x_{n,t+1} - x_{nt}\right) B_1 + \epsilon_{n,t+1,1} - \epsilon_{nt1}\right] \left(\frac{c_{nt+1,1}}{c_{n,t+1,1}}\right)^{\alpha_1}$$

or in logarithmic form:

$$\Delta \ln \lambda_t - \ln \beta = \Delta x_{nt} B_1 + \Delta \epsilon_{nt1} + \alpha_1 \Delta \ln c_{nt1}$$

where:

$$\begin{aligned} \Delta x_{nt} &\equiv \left(x_{n,t+1} - x_{nt}\right) & \Delta \epsilon_{nt1} &\equiv \left(\epsilon_{n,t+1,1} - \epsilon_{nt1}\right) \\ \Delta \ln \lambda_t &\equiv \ln \lambda_{t+1} - \ln \lambda_t & \Delta \ln c_{nt1} &\equiv \ln c_{nt+1,1} - \ln c_{nt1} \end{aligned}$$

- If $E\left[\epsilon_{nt} \left| z_{nt} \right.\right] = 0$ then:

$$E\left\{z_{nt}\left[\Delta \ln \lambda_t - \ln \beta - \alpha_1 \Delta \ln c_{nt1} - \Delta x_{nt} B_1\right]\right\} = 0$$

- A GMM estimator with the usual large sample properties can be formed from the sample analogue.

# Complete Markets

## The rate of time discounting cannot be identified off short panels

- Suppose the model is generated by contingent prices $\lambda_t^e$ and $\beta_0$.
- Assume neither $\lambda_t^e$ nor $\beta_0$ are directly observed.
- Treat $\beta$ and $\Delta \ln \lambda_t$ as parameters in the estimated model. Setting:

$$\begin{aligned} \beta &= \beta^* \\ \Delta \ln \lambda_t^* &= \Delta \ln \lambda_t^e - \ln \beta_0 + \ln \beta^* \end{aligned}$$

- It immediately follows that a specification with:

$$(\beta, \Delta \ln \lambda_t) = (\beta^*, \Delta \ln \lambda_t^*)$$

  is observationally equivalent to:

$$(\beta, \Delta \ln \lambda_t) = (\beta_0, \Delta \ln \lambda_t^e)$$

- Thus the rate of time discounting is not identified off short panels.

# Complete Markets

## The degree of aggregate uncertainty cannot be identified off short panels

- In a world of perfect foresight where the one period interest rate is $i_t$, then:

$$\lambda_t = 1/(1+i_t)$$

and:

$$MRS_{t_n}\left(c_{nt}, c_{n,t+1}\right) \equiv \frac{\beta u_{t_n+1,1}\left(c_{n,t+1}\right)}{u_{t_n,1}\left(c_{nt}\right)} = \frac{1}{(1+i_t)}$$

- We cannot tell from panel data whether this is one of several paths the world could be taking, or whether it is the unique path.
- Confusing "complete markets" with a "full insurance" model, as some authors have done, is misleading.
- Complete markets might fail because of borrowing constraints, rather than a lack of insurance opportunities.

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# An International Comparison

## Descriptive statistics for the U.S. and Germany

| Variables | Year 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 |
|---|---|---|---|---|---|---|---|---|
| Household size | 3.45 | 3.48 | 3.50 | 3.53 | 3.53 | 3.53 | 3.53 | 3.49 |
| | (1.06) | (1.04) | (1.06) | (1.07) | (1.04) | (1.06) | (1.09) | (1.08) |
| Number of children under 16 | 1.10 | 1.09 | 1.04 | 1.03 | 1.01 | .96 | .93 | .89 |
| | (.97) | (.99) | (1.02) | (1.05) | (1.05) | (1.08) | (1.08) | (1.07) |
| Number of rooms | 4.14 | 4.19 | 4.19 | 4.14 | 4.16 | 4.19 | 4.16 | 4.18 |
| | (1.38) | (1.36) | (1.40) | (1.38) | (1.40) | (1.43) | (1.39) | (1.41) |
| Rent[a] | — | 705.92 | 745.59 | 787.36 | 813.38 | 850.08 | 924.48 | 1,002.57 |
| | — | (346.91) | (352.71) | (388.56) | (409.85) | (413.93) | (456.43) | (486.23) |
| Hours worked$^b_m$ | 44.28 | 44.38 | 44.26 | 43.62 | 43.73 | 43.83 | 43.19 | 43.29 |
| | (8.31) | (6.62) | (7.36) | (7.40) | (7.45) | (6.61) | (5.91) | (6.53) |
| Gross labor income$^a_m$ | 3,578.63 | 3,693.23 | 3,936.23 | 4,122.32 | 4,247.84 | 4,434.63 | 4,623.92 | 4,884.37 |
| | (1,170.48) | (1,254.28) | (1,526.49) | (1,632.20) | (1,622.73) | (1,621.31) | (1,712.98) | (1,927.18) |
| Hours worked$^b_f$ | 30.40 | 27.45 | 28.21 | 28.44 | 27.91 | 27.73 | 26.99 | 25.90 |
| | (14.29) | (12.81) | (13.41) | (11.61) | (13.21) | (13.11) | (13.29) | (12.89) |

*Table 2. Descriptive Statistics for the PSID Subsample*

| | Year 1980 | 1981 | 1982 | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 |
|---|---|---|---|---|---|---|---|---|---|
| Household size | 3.98 | 3.92 | 3.87 | 3.82 | 3.84 | 3.83 | 3.78 | 3.83 | 3.82 |
| | (1.65) | (1.60) | (1.52) | (1.40) | (1.42) | (1.42) | (1.26) | (1.25) | (1.23) |
| Number of children under 16 | 1.57 | 1.55 | 1.54 | 1.53 | 1.58 | 1.63 | 1.64 | 1.65 | 1.65 |
| | (1.34) | (1.33) | (1.28) | (1.24) | (1.25) | (1.24) | (1.20) | (1.19) | (1.17) |
| Number of rooms | 5.40 | 5.42 | 5.56 | 5.52 | 5.59 | 5.61 | 5.59 | 5.69 | 5.91 |
| | (1.53) | (1.54) | (1.49) | (1.45) | (1.53) | (1.51) | (1.53) | (1.64) | (1.71) |
| Rent[a] | 221.03 | 241.85 | 266.84 | 273.22 | 301.70 | 322.37 | 334.87 | — | — |
| | (110.76) | (126.11) | (139.86) | (127.97) | (147.72) | (161.70) | (166.73) | — | — |
| Hours worked$^b_m$ | 42.76 | 41.48 | 40.57 | 41.97 | 42.85 | 42.86 | 43.33 | 43.96 | 43.72 |
| | (12.10) | (12.15) | (12.00) | (12.13) | (11.70) | (11.37) | (11.86) | (11.58) | (11.88) |
| Gross labor income$^a_m$ | 1,425.05 | 1,550.07 | 1,605.56 | 1,733.35 | 1,886.44 | 1,957.76 | 2,052.69 | 2,234.64 | 2,334.64 |
| | (838.74) | (946.49) | (1,007.32) | (1,072.58) | (1,337.93) | (1,198.64) | (1,247.06) | (1,456.30) | (1,391.11) |
| Hours worked$^b_f$ | 26.52 | 25.66 | 26.07 | 26.17 | 27.78 | 27.59 | 28.34 | 26.87 | 29.07 |
| | (14.67) | (13.78) | (14.51) | (14.46) | (14.76) | (14.23) | (12.99) | (13.37) | (13.14) |

NOTE: Standard errors are given in parentheses.
[a] Measured on a monthly basis in U.S. dollars.
[b] Measured on a weekly basis in U.S. dollars.
m Variable refers to male.
f Variable refers to female.

# An International Comparison

## A model of male labor supply and housing demand (Miller and Sieg, 1997)

- The following notation applies to household $n$ at time $t$:
  - $l_{0nt}$ female leisure
  - $l_{1nt}$ male leisure
  - $h_{nt}$ housing services
  - $x_{nt}$ observed demographics
  - $(\epsilon_{0nt}, \ldots, \epsilon_{3nt})$ unobserved disturbance *iid* over $n$
- Current utility takes the form:

$$\begin{aligned} u\left(l_{0nt}, l_{1nt}, h_{nt}, , x_{nt}\right) \quad \equiv \quad & \alpha_0^{-1} \exp\left(x_{nt} B_0 + \epsilon_{0nt}\right) h_{nt}^{\alpha_0} l_{0nt}^{\alpha_2} \\ & + \alpha_1^{-1} \exp\left(x_{nt} B_1 + \epsilon_{1nt}\right) l_{1nt}^{\alpha_1} l_{0nt}^{\alpha_3} + \ldots \end{aligned}$$

- The wage rate is the value of the marginal product for a standard labor unit times the efficiency rating of $n$:

$$w_{nt} \equiv w_t \exp\left(x_{nt} B_2 + \epsilon_{2nt}\right)$$

- Similarly:

$$r_{nt} \equiv r_t \exp\left(x_{nt} B_3 + \epsilon_{3nt}\right)$$

# An International Comparison

## Estimates of the marginal rate of substitution functions

| Parameters of utility function | Variable | I | | II | | III | | IV | |
|---|---|---|---|---|---|---|---|---|---|
| | | SOEP | PSID | SOEP | PSID | SOEP | PSID | SOEP | PSID |
| $\alpha_0 - 1$ | | −2.02 | −2.08 | −4.58 | −2.32 | −4.19 | −1.91 | −3.17 | −.91 |
| | | (.22) | (1.13) | (.24) | (.59) | (.26) | (.38) | (1.46) | (1.00) |
| $\alpha_1 - 1$ | | −1.87 | −2.15 | −2.46 | −2.53 | −3.88 | −1.95 | −3.76 | −1.83 |
| | | (.93) | (2.81) | (1.07) | (1.68) | (.73) | (1.22) | (1.51) | (1.97) |
| $\alpha_2 - \alpha_3$ | | −.92 | −1.23 | −.83 | −1.74 | — | — | — | — |
| | | (.89) | (2.81) | (1.04) | (1.67) | | | | |
| $\alpha_2$ | | — | — | — | — | −.31 | −.66 | −.29 | −.61 |
| | | | | | | (.84) | (2.14) | (3.00) | (4.20) |
| $\alpha_3$ | | — | — | — | — | 2.09 | .47 | 2.29 | .56 |
| | | | | | | (.39) | (2.11) | (2.19) | (2.58) |
| $\Delta B$ | Household size | −.21 | −.16 | −.46 | .50 | — | — | — | — |
| | | (.20) | (.26) | (.21) | (.36) | | | | |
| | Number of children | .07 | .10 | .29 | −.15 | — | — | — | — |
| | | (.22) | (.30) | (.12) | (.29) | | | | |
| $B_0$ | Household size | — | — | — | — | .33 | .39 | .40 | .15 |
| | | | | | | (.26) | (.48) | (.58) | (1.01) |
| | Number of children | — | — | — | — | −.09 | −.11 | −.23 | −.05 |
| | | | | | | (.20) | (.53) | (.81) | (.96) |
| $B_1$ | Household size | — | — | — | — | .02 | .13 | .13 | .11 |
| | | | | | | (.21) | (.31) | (.51) | (.44) |
| | Number of children | — | — | — | — | .22 | −.15 | −.13 | −.12 |
| | | | | | | (.17) | (.46) | (.64) | (.42) |
| $B_3$ | Size of housing unit | .28 | .19 | .47 | .27 | .48 | .37 | .41 | .27 |
| | | (.03) | (.08) | (.02) | (.06) | (.02) | (.06) | (.11) | (.16) |
| | City indicator | .76 | .52 | −.11 | .63 | .94 | .55 | 1.37 | .54 |
| | | (.05) | (.24) | (.04) | (.15) | (.05) | (.02) | (.39) | (.23) |
| | $J$ value | 197.74 | | 333.48 | | 331.46 | | 242.42 | |
| | Degrees of freedom | 207 | | 319 | | 321 | | 209 | |

# An International Comparison

## Interpreting Table 3

- The column key is:
  1. MRS between housing and male leisure plus housing rental function
  2. Adds wage equation
  3. Adds intertemporal MRS for male leisure over consecutive periods and subtracts rent equation
  4. Both MRS conditions plus wage and rent equations
- The number of observations is about 400 so $\sqrt{N}$ is about 20.
- $J$ is asymptotically $\chi_d$ where $d = \#$ overidentifying restrictions.
- None of the specifications is rejected, all the coefficients are significant and are signed according to economic intuition.
- Contingent claims prices (inversely) track aggregate consumption quite well.

# An International Comparison

Aggregate consumption (solid line) and estimated contingent prices (dotted) for Germany and the U.S.

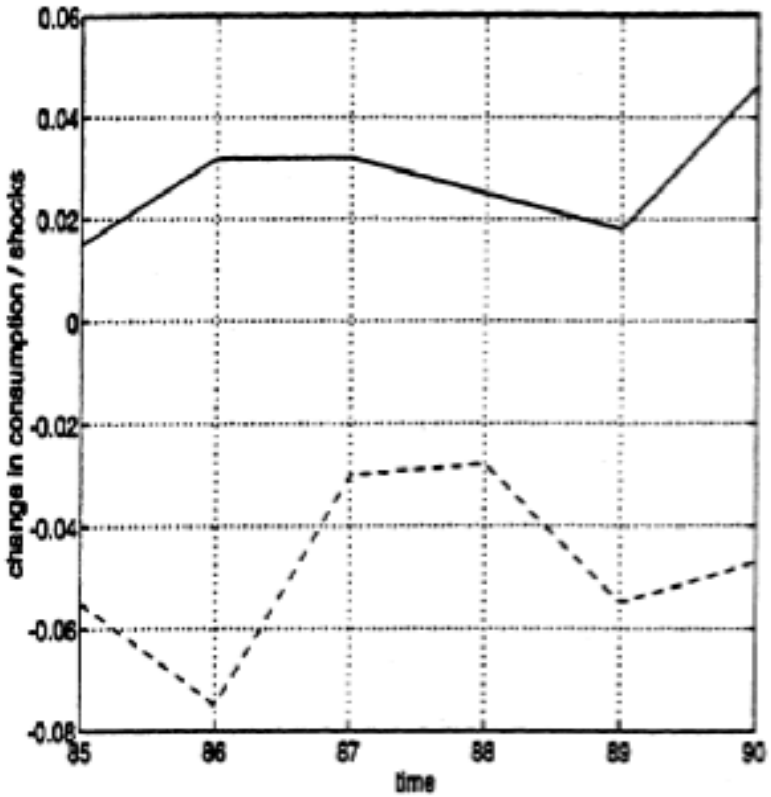



Figure I.Aggregate Consumption and Shocks in Germany

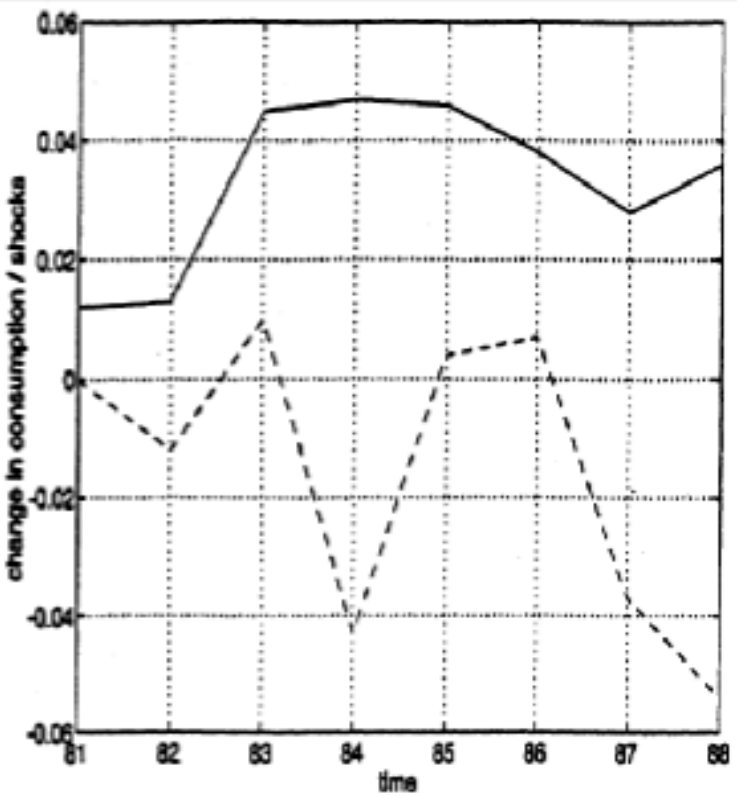



Figure 2. Aggregate Consumption and Shocks in the US

# An International Comparison

## Testing equality of prices, preferences and efficiency ratings

- We reject the null hypotheses that:
  - contingent claims prices between Germany and US are equal
  - contingent claims prices between different regions in the US are equal at the 0.05 but not at the 0.1 level
  - preferences between the two countries are the same
- With respect to purchasing power parity we:
  - do not reject the null that the value of marginal product of labor is equalized across both countries
  - reject the null that the premium to education is the same.

# Portfolio Choices in Competitive Equilibrium

## Relaxing the complete markets assumption

- Suppose there are $J$ financial securities.
- Let $p_{tj}$ denote the price of the $j^{th}$ security in period $t$ consumption units, and $q_{t-1,j}$ the amount a consumer owns at the beginning of the period.
- Let $r_{tj}$ denote the real return on assets purchased in period $t-1$.
- The investor's budget constraint is:

$$c_t + \sum_{j=1}^{J} p_{tj} q_{tj} \leq \sum_{j=1}^{J} r_{tj} p_{t-1,j} q_{t-1,j}$$

- At $t$ the consumer maximizes a concave objective function with linear constraints, choosing $(q_{s1}, ..., q_{sJ})$ to maximize:

$$u(c_t) + E_t \left[ \sum_{s=t+1}^{T} \beta^{s-t} u(c_s) \right]$$

subject to the sequence of all the future budget constraints.

# Portfolio Choices in Competitive Equilibrium

## First order conditions

- Nonsatiation guarantees:

$$c_t = \sum_{j=1}^{J} \left( r_{tj} p_{t-1,j} q_{t-1,j} - p_{tj} q_{tj} \right)$$

- The interior first order condition for each $k \in \{1, \ldots, J\}$ requires:

$$\begin{aligned} & p_{tk} u' \left( \sum_{j=1}^{J} \left( r_{tj} p_{t-1,j} q_{t-1,j} - p_{tj} q_{tj} \right) \right) \\ \geq \quad & E_t \left[ p_{tk} r_{t+1,k} \beta u' \left( \sum_{j=1}^{J} \left( r_{t+1,j} p_{tj} q_{tj} - p_{t+1,j} q_{t+1,j} \right) \right) \right] \end{aligned}$$

with equality holding if $q_{tj} > 0$.

# Portfolio Choices in Competitive Equilibrium

## The fundamental theorem of portfolio choice

- Substituting $c_t$ and $c_{t+1}$ back into the marginal utilities and rearranging yields the fundamental equation of portfolio choice:

$$1 = E_t\left[r_{t+1,k}\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right] \equiv E_t\left[r_{t+1,k}MRS_{t+1}\right]$$

- Recall from the definition of a covariance:

$$\begin{aligned} cov\left(r_{t+1,k}, MRS_{t+1}\right) &= E_t\left[r_{t+1,k}MRS_{t+1}\right] - E_t\left[r_{t+1,k}\right]E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right]E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right]/r_{t+1} \end{aligned}$$

  where the second line uses the fundamental equation of portfolio choice, and the third the definition of the risk free rate.

- Rearranging this equation gives the risk correction for the $k^{th}$ asset:

$$E_t\left[r_{t+1,k}\right] - r_{t+1} = -r_{t+1}cov\left(r_{t+1,k}, MRS_{t+1}\right)$$

# Portfolio Choices in Competitive Equilibrium

## Estimation and testing (Hansen and Singleton, 1982)

- For any $r \times 1$ vector $x_t$ belonging to the information set at $t$ and all $k$:

$$0 = E_t \left[ r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \right] = E \left[ r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \,|\, x_t \right]$$

and hence:

$$0 = E \left\{ x_t \left[ r_{t+1,k} \beta \frac{u'(c_{t+1})}{u'(c_t)} - 1 \right] \right\}$$

- Given a sample of length $T$ we can estimate the $1 \times l$ vector $(\beta, \alpha)$ for a parametrically defined utility function $u(c_t; \alpha)$ by solving:

$$0 = A_T \sum_{t=1}^{T} x_t \left[ r_{t+1,k} \beta \frac{u'(c_{t+1}; \alpha)}{u'(c_t; \alpha)} - 1 \right]$$

where $A_T$ is an $l \times r$ weighting matrix.

- Clearly this estimator easily generalizes to any number of assets with an interior condition.

- For example if lifetime utility is:

$$\sum_{t=1}^{\infty} \beta^t u(c_t) = (1+\alpha)^{-1} \sum_{t=1}^{\infty} \beta^t c_t^{1+\alpha}$$

then:

$$0 = E\left\{ x_t \left[ r_{t+1,k} \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha} - 1 \right] \right\}$$

# Portfolio Choices in Competitive Equilibrium

Estimates from aggregate consumption data (Hansen and Singleton, 1984, Table I)

| Cons | Return | *NLAG* | $\hat{\alpha}$ | $\widehat{SE}(\hat{\alpha})$ | $\hat{\beta}$ | $\widehat{SE}(\hat{\beta})$ | $\chi^2$ | DF | Prob |
|---|---|---|---|---|---|---|---|---|---|
| NDS | EWR | 1 | −0.9360 | 2.5550 | .9930 | .0060 | 5.226 | 1 | .9774 |
| NDS | EWR | 2 | 0.1529 | 2.3468 | .9906 | .0056 | 7.378 | 3 | .9392 |
| NDS | EWR | 4 | 1.2605 | 2.2669 | .9891 | .0059 | 9.146 | 7 | .7577 |
| NDS | EWR | 6 | 0.1209 | 2.0455 | .9928 | .0054 | 14.556 | 11 | .7963 |
| NDS | VWR | 1 | −1.0350 | 1.8765 | .9982 | .0045 | 1.071 | 1 | .6993 |
| NDS | VWR | 2 | 0.1426 | 1.7002 | .9965 | .0044 | 3.467 | 3 | .6749 |
| NDS | VWR | 4 | −0.0210 | 1.6525 | .9969 | .0043 | 5.718 | 7 | .4270 |
| NDS | VWR | 6 | −1.1643 | 1.5104 | .9997 | .0041 | 11.040 | 11 | .5601 |
| ND | EWR | 1 | −1.5906 | 1.0941 | .9930 | .0034 | 7.186 | 1 | .9926 |
| ND | EWR | 2 | −0.7127 | 0.9916 | .9918 | .0034 | 12.040 | 3 | .9928 |
| ND | EWR | 4 | −0.1261 | 0.8917 | .9921 | .0035 | 14.638 | 7 | .9591 |
| ND | EWR | 6 | −0.4193 | 0.8256 | .9936 | .0033 | 18.016 | 11 | .9188 |
| ND | VWR | 1 | −1.2028 | 0.7789 | .9976 | .0027 | 1.457 | 1 | .7726 |
| ND | VWR | 2 | −0.5761 | 0.7067 | .9975 | .0027 | 5.819 | 3 | .8792 |
| ND | VWR | 4 | −0.6565 | 0.6896 | .9978 | .0027 | 7.923 | 7 | .6606 |
| ND | VWR | 6 | −0.9638 | 0.6425 | .9985 | .0027 | 10.522 | 11 | .5159 |

# Portfolio Choices in Competitive Equilibrium

## Estimates from aggregate consumption data (Hansen and Singleton, 1984, Table III)

Equally- and Value-Weighted Aggregate Returns 1959:2–1978:12

| Cons | NLAG | $\hat{\alpha}$ | $\widehat{SE}(\hat{\alpha})$ | $\hat{\beta}$ | $\widehat{SE}(\hat{\beta})$ | $\chi^2$ | DF | Prob. |
|---|---|---|---|---|---|---|---|---|
| NDS | 1 | −0.5901 | 1.7331 | .9989 | .0041 | 18.309 | 6 | .9945 |
| NDS | 2 | 1.0945 | 1.4907 | .9961 | .0040 | 24.412 | 12 | .9821 |
| NDS | 4 | 0.3835 | 1.4208 | .9975 | .0039 | 40.234 | 24 | .9798 |
| ND | 1 | −0.6494 | 0.6838 | .9982 | .0025 | 19.976 | 6 | .9972 |
| ND | 2 | −0.0200 | 0.6071 | .9982 | .0025 | 27.089 | 12 | .9925 |
| ND | 4 | −0.1793 | 0.5928 | .9986 | .0025 | 42.005 | 24 | .9871 |

Value-Weighted Aggregate Stock Returns and Risk-Free Bonds Returns 1959:2–1978:12

| Cons | NLAG | $\hat{\alpha}$ | $\widehat{SE}(\hat{\alpha})$ | $\hat{\beta}$ | $\widehat{SE}(\hat{\beta})$ | $\chi^2$ | DF | Prob. |
|---|---|---|---|---|---|---|---|---|
| NDS | 1 | −.1405 | .0420 | .9998 | .0001 | 31.800 | 8 | .9999 |
| NDS | 2 | −.1472 | .0376 | .9998 | .0001 | 44.083 | 16 | .9998 |
| NDS | 4 | −.1405 | .0320 | .9996 | .0001 | 65.250 | 32 | .9995 |
| ND | 1 | −.0962 | .0461 | .9995 | .0001 | 25.623 | 8 | .9988 |
| ND | 2 | −.1150 | .0377 | .9995 | .0001 | 39.874 | 16 | .9991 |
| ND | 4 | −.1611 | .0364 | .9994 | .0001 | 60.846 | 32 | .9985 |

Three Industry-Average Stock Returns 1959:2–1977:12

| Cons | NLAG | $\hat{\alpha}$ | $\widehat{SE}(\hat{\alpha})$ | $\hat{\beta}$ | $\widehat{SE}(\hat{\beta})$ | $\chi^2$ | DF | Prob. |
|---|---|---|---|---|---|---|---|---|
| NDS | 1 | 1.5517 | 1.8006 | .9906 | .0046 | 13.840 | 13 | .6147 |
| NDS | 4 | 0.6713 | 1.2466 | .9940 | .0035 | 88.211 | 49 | .9995 |
| ND | 1 | 0.7555 | 0.7899 | .9924 | .0029 | 13.580 | 13 | .5959 |
| ND | 4 | 0.5312 | 0.5512 | .9939 | .0024 | 89.501 | 49 | .9996 |

# Portfolio Choices in Competitive Equilibrium

## Interpreting estimates from aggregate data

- To interpret these results:
  - NDS (nondurables plus services)
  - ND (nondurables)
  - EWR (NYSE equally weighted average returns)
  - VWR (NYSE value weighted average returns)
  - The 3 industries are chemicals, transportation and equipment, and other retail.
- Note that:
  - 10 out of 12 specifications in Table III are rejected at the 0.05 level.
  - In the other 2 specifications in Table III $\alpha > 0$, which implies $u(c_t)$ convex increasing (so the FOC is not a maximum).

# Portfolio Choices in Competitive Equilibrium

## Applying time series data to an individual consumer investor

- Subscript current utility, consumption and instruments by individual $n \in \{1, \ldots, N\}$ and define the forecast error $\epsilon_{nt}$ as:

$$\epsilon_{nt} \equiv \left[ r_{tk} \beta u_n' \left( c_{t+1} \right) / u_n' \left( c_t \right) \right] - 1$$

- Then if the FOC asset for $k$ holds with equality for $n$:

$$E_t \left[ \epsilon_{nt} \left| x_{nt} \right. \right] = 0 \Longrightarrow E \left[ \epsilon_{nt} \left| x_{nt} \right. \right] = 0 \Longrightarrow E \left[ x_{nt} \epsilon_{nt} \right] = 0$$

- Noting $\{\ldots, \epsilon_{nt}, \epsilon_{n,t+1}, \ldots\}$ is a Martingale difference sequence, the Euler equation approach to estimation could be applied, since for all $n \in \{1, \ldots, N\}$:

$$0 = E \left[ x_{nt} \epsilon_{nt} \right] = \underset{T \to \infty}{p \lim} \left[ \frac{1}{T} \sum_{t=1}^{T} x_{nt} \epsilon_{nt} \right]$$

- The large sample properties of this estimator rely on the length of the time series, or more intuitively, the fact that successive forecast errors are uncorrelated with each other.

# Relaxing the Assumption of a Representative Consumer

## Using Euler equation methods on a cross section

- Suppose $u_n(c_t) = u(c_t)$ and the data only comprise a small number of periods (say 2), but a large number of individuals.
- Averaging over $n \in \{1, \ldots, N\}$ gives:

$$\frac{1}{N}\sum_{n=1}^{N} x_{nt}\left[r_{tk}\beta\frac{u'(c_{t+1})}{u'(c_t)} - 1\right] = \frac{1}{N}\sum_{n=1}^{N} x_{nt}\epsilon_{nt}$$

- Assuming each person in the sample lived in different disconnected economy from everyone, implying the forecast error $\epsilon_{nt}$ for $n$ is independent of $\epsilon_{n't}$ for $n'$:

$$\operatorname*{plim}_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N} x_{nt}\epsilon_{nt}\right] = 0 = \operatorname*{plim}_{T\to\infty}\left[\frac{1}{T}\sum_{t=1}^{T} x_{nt}\epsilon_{nt}\right]$$

- Thus $u(c_t)$ could be identified and estimated off a panel.

# Relaxing the Assumption of a Representative Consumer

## Why Euler equation estimation methods fail on a cross section (Altug and Miller, 1990)

- Now suppose they live in the same economy. Then:

$$v_t \equiv \operatorname*{plim}_{N \to \infty} \left[ \frac{1}{N} \sum_{n=1}^{N} x_{nt} \epsilon_{nt} \right] \neq 0$$

- In this case $v_t$ depends on any aggregate shock to the economy hitting everyone in the economy.
- Furthermore $v_t$ is instrument specific, so treating $v_t$ as a time dummy in estimation requires as many time dummies as there are instruments.
- Hence $u(c_t)$ is not identified off a panel formed from individuals who are subjected to correlated disturbances over time.

# Portfolio Choices in Competitive Equilibrium

## Heterogeneity with complete markets

- Now suppose $z_{nt}$ is a vector differentiating consumers:

$$u_{nt}\left(c_{nt}\right) \equiv \left(\alpha+1\right)^{-1} \delta\left(z_{nt}\right) c_{nt}^{\alpha+1} \tag{7}$$

- Then (6) simplifies to:

$$\beta^{t_n} \delta\left(z_{nt}\right) c_{nt}^{\alpha} = \eta_n \lambda_t$$

- Hence the asset pricing equation can be expressed as:

$$1 = \beta E_t\left[r_{t+1,k} \frac{\delta\left(z_{n,t+1}\right)}{\delta\left(z_{nt}\right)} \left(\frac{c_{n,t+1}}{c_{nt}}\right)^{\alpha}\right] = E_t\left[r_{t+1,k} \frac{\lambda_{t+1}}{\lambda_t}\right] \tag{8}$$

- Solving for consumption and averaging over a population of $N$ yields:

$$c_t \equiv \sum_{n=1}^{N} c_{nt} = \lambda_t^{\frac{1}{\alpha}} \sum_{n=1}^{N} \left[\eta_n / \beta^{t_n} \delta\left(z_{nt}\right)\right]^{\frac{1}{\alpha}}$$

or the shadow value of consumption:

$$\lambda_t = c_t^{\alpha} \left\{\sum_{n=1}^{N} \left[\eta_n / \beta^{t_n} \delta\left(z_{nt}\right)\right]^{\frac{1}{\alpha}}\right\}^{-\alpha}$$

# Portfolio Choices in Competitive Equilibrium

## Econometric implications of heterogeneity in CRRA preferences with complete markets

- Substituting the expression for $\lambda_t$ into (8) gives: the Euler equation for a representative consumer:

$$1 = \beta E_t \left[ r_{t+1,k} \left\{ \frac{\sum_{n=1}^{N} \left[ \eta_n / \beta^{t_n} \delta \left( z_{nt} \right) \right]^{\frac{1}{\alpha}}}{\sum_{n=1}^{N} \left[ \eta_n / \beta^{t_n+1} \delta \left( z_{n,t+1} \right) \right]^{\frac{1}{\alpha}}} \right\}^{\alpha} \left( \frac{c_{t+1}}{c_t} \right)^{\alpha} \right]$$

1. If consumers are only differentiated by some fixed characteristics and their initial wealth, and $z_{nt} \equiv z_n$ then:

$$\beta \left( c_{t+1} / c_t \right)^{\alpha} = \lambda_{t+1} / \lambda_t$$

2. However to achieve a representative consumer style portfolio equation if $z_{nt}$ changes over time, returns must be reweighted to reflect shifting priorities for consumption in $t$ by different segments of the population.

- Therefore one reason for rejecting the representative consumer model is time varying heterogeneity.

# Portfolio Choices in Competitive Equilibrium

## A third problem with this approach: predicting counterfactuals

- We have already mentioned two problems with this approach:
  1. Appealing to large sample properties when there are only a few hundred observations is questionable here. (Consistently accurate short term macro predictions are hard.)
  2. Observed heterogeneity in the population is typically inconsistent with the representative consumcer paradigm.
- There is a third problematic issue:
  - In order to solve the model we must lay out a set of state variables.
  - These state varialves would define how information evolves.
  - This would yield exact consumption and asset pricing rules.
  - The deterministic predictions are bound to be contradicted by the data.
  - The framework does not provide a measure of the discrepancy.
  - Hence predictions about counterfactuals are speculative.