Portfolio Investment

Robert A. Miller

Trade and Investment Strategy

December 2026

Introduction

A framework for analysing portfolio choice

- *Competitive equilibrium* is commonly assumed in portfolio analysis, widely used to finesse descriptions of institutional detail:
  1. Markets are *perfectly liquid*. (Bid equals ask)
  2. Individual order size does not affect prices. (*Price taking* behavior)
  3. There are neither unfilled orders nor excess inventories. (*Markets clear*)
- The first two conditions of competitive equilibrium imply that investors are price takers, and are most plausible for small investors.
- Large trades typically affect the spread, and therefore **block traders** should also account for the effects of their trading activity on prices they pay.
- About half the trading in financial securities of publicly listed firms are large trades (over $200,000), many executing off the limit order books.

A Model of Portfolio Choice

Assets

- We now derive the fundamental equation that characterizes:
  - the trade-off at the individual level between current and future consumption in terms of asset returns
  - a **personalized mean variance frontier** for small investors.
- Suppose there are J financial securities. Denote by:
  - q_{tj} the amount of the j^{th} security owned at the end of period t.
  - p_{tj} the price of (buying or selling) the j^{th} security in period t, measured in consumption units.
  - π_{tj} the asset growth in the j^{th} security revealed at the beginning of period t and applied to the securities carried from the previous period.
- Holding q_{tj} units of j at the end of t produces $\pi_{t+1,j} q_{tj}$ when period $t+1$ begins.

A Model of Portfolio Choice

Law of motion for wealth

- Individuals also receive wages at the beginning of each period t.
- Let w_t denote real wages (and other income).
- Upon entering the current period, the individual receives a return on assets she holds, receives a wage, trades, consumes, and carries some assets and cash over to the next period.
- The *law of motion for her wealth* is:

$$\begin{aligned} c_t + \sum_{j=1}^{J} p_{tj} q_{tj} &= w_t + \sum_{j=1}^{J} p_{tj} \pi_{tj} q_{t-1,j} \qquad (1) \\ c_T &= \sum_{j=1}^{J} p_{Tj} \pi_{Tj} q_{T-1,j} \end{aligned}$$

where T is the last period of of her life.

A Model of Portfolio Choice

Maximization problem

- The individual makes choices $(q_{t1}, ..., q_{tJ})$ each period t to maximize her expected lifetime utility:

$$E\left[\sum_{t=1}^{T}\beta^{t}u\left(c_{t}\right)\right]=E\left\{\sum_{t=1}^{T}\beta^{t}u\left[w_{t}+\sum_{j=1}^{J}p_{tj}\left(\pi_{tj}q_{t-1,j}-q_{tj}\right)\right]\right\} \tag{2}$$

  subject to a solvency condition (1) where $u(\cdot)$ is a concave increasing function.

- Notice that q_{tk} only enters (2) in two spots, namely:

$$u\left(w_{t}+\sum\nolimits_{j=1}^{J}p_{tj}\left(\pi_{tj}q_{t-1,j}-q_{tj}\right)\right)$$

$$\text{and } u\left(w_{t+1}+\sum\nolimits_{j=1}^{J}p_{t+1,j}\left(\pi_{t+1j}q_{tj}-q_{t+1,j}\right)\right)$$

Optimal Choices

The first order condition

- The *interior first order condition* (FOC) for $k \in \{1, \ldots, J\}$ is:

$$p_{tk} u' \left(w_t + \sum_{j=1}^{J} p_{tj} \left(\pi_{tj} q_{t-1,j} - q_{tj} \right) \right) \tag{3}$$

$$= E_t \left[p_{t+1,k} \pi_{t+1,k} \beta u' \left(w_{t+1} + \sum_{j=1}^{J} p_{t+1,j} \left(\pi_{t+1,j} q_{tj} - q_{t+1,j} \right) \right) \right]$$

  or, upon substiuting back in for consumption:

$$p_{tk} u' \left(c_t \right) = E_t \left[p_{t+1,k} \pi_{t+1,k} \beta u' \left(c_{t+1} \right) \right] \tag{4}$$

- If no units of the k^{th} asset are held, that is $q_{tk} = 0$ then:

$$u' \left(c_t \right) > E_t \left[r_{t+1,k} \beta u' \left(c_{t+1} \right) \right]$$

Then the marginal utility of current consumption $u' \left(c_t \right)$ is too high to justify acquiring k given the expected benefits of its return on consumption next period.

Optimal Choices

The fundamental equation of portfolio choice

- Define the real rate of return on asset k at the beginning of t by:

$$r_{tk} \equiv \frac{p_{tk}\pi_{tk}}{p_{t-1,k}}$$

- Dividing both sides of (4) by $p_{tk}u'(c_t)$ yields:

$$1 = E\left[\left(\frac{p_{2k}\pi_{2k}}{p_{1k}}\right)\beta\frac{u'(c_2)}{u'(c_1)}\right] \equiv E_t\left[r_{2k}MRS(c_{t+1}, c_t)\right] \qquad (5)$$

which is called the **fundamental equation of portfolio choice**.

- Note how similar (3) is to the FOCs for analyzing current and future consumption (analyzed in the previous lecture).
  - If there was no uncertainty then $r_{tk} = 1 + i_t$ and (5) reduces to:

$$1 = (1 + i_t)\,MRS(c_{t+1}, c_t)$$

Optimal Choices

Utility with constant relative risk aversion

- When $u(c_t) = c_t^{1-\alpha} / (1-\alpha)$ the equation yields:

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_t}{c_{t+1}} \right)^{\alpha} \right] = 1$$

- For example, in a world where there is no uncertainty, and r_{t+1} is the risk free rate, this equation simplifies to:

$$\beta r_{t+1} \left(\frac{c_t}{c_{t+1}} \right)^{\alpha} = 1$$

- And specializing to $u(c_t) = \ln(c_t)$ we obtain:

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_t}{c_{t+1}} \right) \right] = 1$$

- Finally if $u(c_t) = c_t$ the expected return is equated with the interest rate for all interior choices:

$$\beta E_t \left[r_{t+1,k} \right] = 1$$

Rebalancing

- Define W_t, disposable wealth (including wage income) at the beginning of period t as:

$$W_t \equiv w_t + \sum_{j=1}^{J} p_{tj} \pi_{tj} q_{t-1,j}$$

- Then (3) can be expressed as:

$$u'\left(W_t - \sum_{j=1}^{J} p_{tj} q_{tj}\right) = E_t\left[r_{t+1,k} \beta u'\left(W_{t+1} - \sum_{j=1}^{J} p_{t+1,j} q_{t+1,j}\right)\right]$$

- Given W_t this optimality condition does not depend on the performance of individual assets up until period t.
- Transaction fees (plus human capital and insider information) aside, current investment decision should not depend on the past.

For example if $r_{tk} > r_{tk'}$, then you should typically sells units of k and buy units of k'.

- This is called **rebalancing**

Optimal Choices

Risk correction

- Recall from the definition of a covariance:

$$\begin{aligned} cov\left(r_{t+1,k}, MRS_{t+1}\right) &= E_t\left[r_{t+1,k} MRS_{t+1}\right] - E_t\left[r_{t+1,k}\right] E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right] E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right] / r_{t+1} \end{aligned}$$

  where:

  - the second line uses the fundamental equation of portfolio choice
  - the third line uses the definition of the risk free rate.

- Rearranging this equation gives the risk correction for the k^{th} asset:

$$E_t\left[r_{t+1,k}\right] - r_{t+1} = -r_{t+1} cov\left(r_{t+1,k}, MRS_{t+1}\right)$$

  where $E_t\left[r_{t+1,k}\right] - r_{t+1}$ is the **expected excess return**.

Optimal Choices

Risk correction

- Dividing through by r_{t+1}:

$$\frac{E_t\left[r_{t+1,k}\right] - r_{t+1}}{r_{t+1}} = -cov\left(r_{t+1,k}, MRS_{t+1}\right)$$

The expected excess return on an asset as a proportion of the risk free rate equals the negative of the covariance between the asset return and the marginal rate of substitution between current and future consumption.

- When c_{t+1} and $r_{t+1,k}$ tend to move in opposite directions (like insurance payouts), then $cov\left(r_{t+1,k}, MRS_{t+1}\right)$ is positive and the investor accepts a mean return less than the risk free rate.
- When they mainly move together (like the market portfolio and wages in many occupations), then $cov\left(r_{t+1,k}, MRS_{t+1}\right)$ is negative and he requires more.

Financial Intermediation

Mutual Funds

- If trading was a costless exercise, then investing through a mutual fund would be free.
- Consider investing through a personally tailored mutual fund which comprises the n^{th} investor's portfolio:

$$s_t^{(n)} = \sum_{j=1}^{J} p_{tj} q_{tj}^{(n)}$$

- Its real return is defined by:

$$\pi_{t+1,s} \equiv \left(\sum_{j=1}^{J} p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j} \right) / s_t^{(n)}$$

- The individual could economize on his transactions by purchasing shares in a mutual fund that replicate the financial portfolio he would have purchased himself.

Financial Intermediation

Mutual funds also satisfy the fundamental equation of portfolio choice

- Weighting the first order condition by $q_{tj}^{(n)}$ and then summing over j gives:

$$u'(c_t)\, s_t^{(n)} = \sum_{j=1}^{J} p_{tj} q_{tj}^{(n)} u'(c_t) = \sum_{j=1}^{J} E_t \left[p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j} \beta u'(c_{t+1}) \right]$$

- Dividing both sides of this equation by $u'(c_t)\, s_t^{(n)}$ yields the fundamental equation of portfolio choice for the mutual fund:

$$1 = E_t \left[\sum_{j=1}^{J} \left(\frac{p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j}}{s_t^{(n)}} \right) \beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \equiv E_t \left[r_{t+1,s} MRS_{t+1} \right]$$

- There are no tax implications. Rather than buying a whole portfolio of stocks, the investor simply picks the mutual with a return that most suits his marginal rate of substitution at the same time he decides how much to consume in the current period.

Financial Intermediation

Why don't individuals form their own portfolios?

- Since transaction fees per share decline with volume, and information about the risk characteristics of each stock is costly, investors with like minded strategies have an incentive to band together. Mutual funds also give investors a say on the board.
- Recall that with the same wealth and wage process (plus other demographics) investors with same utility function should hold the same portfolio. Thus mutual funds compete with each other by constructing portfolios that appeal to different niches in the market.
- This explains why the S & P composite of 1500 firms covers about 85 percent of the U.S. equity market, there are more than 8000 mutual funds to invest in.

Financial Intermediation

Hedges

- Many companies routinely hedge on input prices, and insure themselves against financial events they cannot control.
- For example fuel is a very large operating cost for airlines:
  - Many long distance carriers hedge against fluctuations in fuel prices (British Airways, Singapore Airlines, Qantas) buy oil futures.
  - Delta even owns and operates its own oil refinery (hedging with real assets).
  - Southwest used ot hedge against unanticipated changes in fuel prices.
- Companies cannot hedge risks or insure themselves against negative financial events they influence, because this would reduce their incentives to avoid the risk, creating a moral hazard problem.
- But why should companies hedge any risks, since individual investors and the financial retail sector can replicate every hedging strategy for themselves?

Financial Intermediation

Why do companies hedge?

- There are probably two main reasons:
  1. If there are transactions costs and information imperfections, hedging and insurance reduces distractions, helps solidify the company's mission and branding strategy, makes the company's performance more dependent on its core business, thus increasing its transparency and value as a financial security.
  2. Liquidity flow concerns can also be an issue. (For example, traditionally business schools have relied on hybrid/part-time programs in the economic booms and full-time programs in recessions.)
- When dealing with fluctuations in the price of fuel:
  - long distance carriers (Qantas) may be a little less flexible in their flight schedules than domestic carriers (Southwest).
  - Delta does both . . . so perhaps uses a heavier hammer to nail down a solution.

Summary

Optimal portfolio choices

- Explicitly solving for the bid ask spread in an limit order exchange is too difficult even with supercomputer help.
- Competitive equilibrium (pricing taking in perfectly liquid markets) greatly simplifies portfolio analysis.
- Optimal behavior requires the investor to discount, relative to the real interest rate, stocks with returns (like aggregate shocks to the economy) that are positively correlated with his own next period consumption choices.
- Similarly an investor should favor stocks over bonds when (like insurance) their returns help oset variation in his consumption next period.

Summary

Financial intermediation

- Market equilibrium does not require investors to hold the same portfolio, or even a mix of a small number of funds.
- However the optimal financial portfolio holdings of each investor is independent of their initial endowment unless there is a nontradeable component to their endowment that is correlated with a traded asset.
- Mutual funds and hedging by firms are two illustrations of the same phenomenon, financial intermediation.
- Financial intermediaries bundle securities for investors to tailor securities that suit the investor's attitudes towards current versus future consumption and certain versus risky consumption.
- They also provide hedging insurance to investors and companies exposed to risks that are ancillary to their mission and tastes.